Elementary Calculus

Week 1

This week we will review the definition, meaning and rules for the derivative of a function. We will also look at implicit differentiation and tangent line calculations. KEYWORDS: derivative (as slope of tangent, as rate of change, of a product, etc.), differentiation (formulas, implicit), tangent line(to a curve)

Technique Time

[1] Find $\frac{dy}{dx}$ for each of the following:

(a) $y = \frac{1}{x-2}$ (b) $y = (3x^2+1)^4$ (c) $y = \sqrt{x^2+9}$ (d) $y = \frac{x^2}{\sqrt{x^2+9}}$

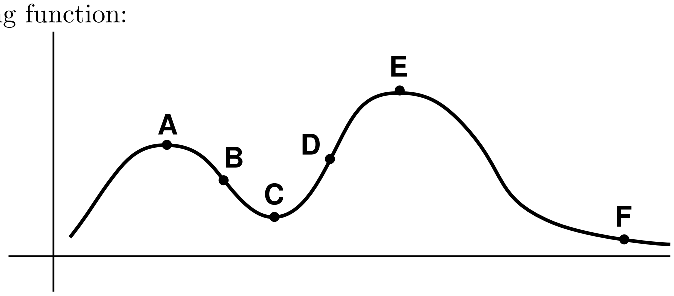
(e) $y = \sqrt{\frac{2x}{x+1}}$ (f) $y = (3x^2-7)\sqrt{x}$.

[2] Suppose that f and g are differentiable functions and that:

$$f(3) = 4, \quad f'(2) = 6, \quad g(3) = 2, \quad g(2) = 7, \quad g'(3) = 7, \quad g'(2) = 3.$$

If $y = f(g(x))$, what is the value of $\frac{dy}{dx}$ at $x = 3$?

[3] Consider the following function:



At which of the labelled points does the function have: (i) the maximum rate of change?

(ii) a tangent with minimum slope?

(iii) a derivative with value 0 ?

(iv) a local maximum?

[4] Use the definition to calculate a formula for the derivative of each of the following functions:

(a) x^3 (b) $\sqrt{x}$ (c) $x + \frac{1}{x}$

[5] Calculate $\frac{dy}{dx}$ given that:

(a) $x^2 + y^2 = 7$ (b) $x^2 + xy - y^3 = xy^2$ (c) $\sqrt{x} + \sqrt{y} = 25$

[6] Find the equations of the tangent lines to the following curves at the specified points:

(a) $x^3 + y^3 = 9$ at $(2, 1)$ (b) $xy^2 = a^2$ at $(1, a)$ (c) $y^2 = \frac{x^2}{xy-4}$ at $(4, 2)$

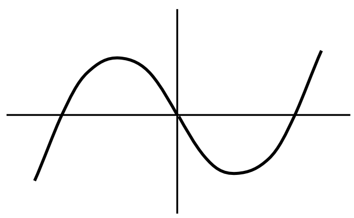
[7] Show that the sum of the x-intercept and the y-intercept of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is always c.

[8] Assume that $f(x)$ is continuous and differentiable near $x=c$. Find

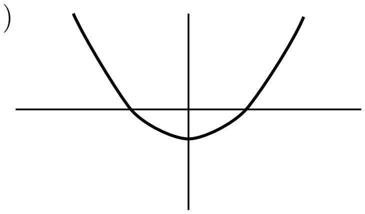
$$\lim_{h\to 0}\frac{f(c+2h)-f(c+h)}{h}.$$

[9] There follow two sets of three graphs arranged vertically. In each trio there appears the graph of a function, of its derivative and of its second derivative. Your mission is to identify, in each case, which is the function, which the derivative and which the second derivative.

(a)



(b)



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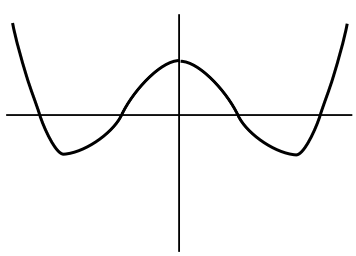

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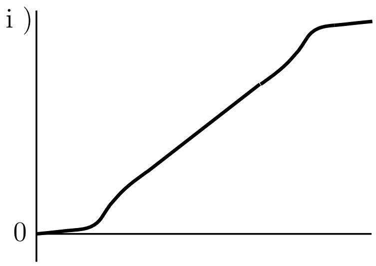

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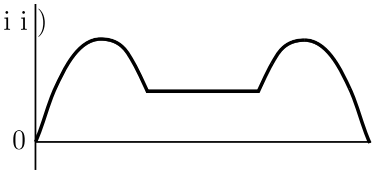

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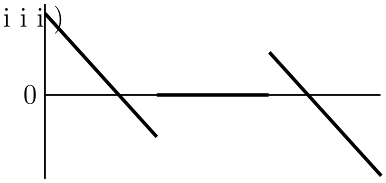


Elementary Calculus

Week 2

This week we will look at derivatives of trigonometric functions, Newton's Method and L'Hôpital's rule.

KEYWORDS: Trigonometric functions (derivatives of), trigonometric identities, Newton's method, L'Hospital's rule.

TECHNIQUE TIME

[1] Calculate $\frac{dy}{dx}$ for:

(a) $y = \sin^2(x)$ (b) $y = \cos(5x^3 + 2)$ $y = 2\sin x + \tan^2(x)$.

[2] Calculate $\frac{dy}{dx}$ for:

(a) $y = x \sin x$ (b) $y = (\cos x)\sqrt{x^2+3}$ (c) $y = \cos(x\sqrt{x^2+3})$.

[3] Determine the equation of the tangent to the curve $y = \tan x + \sec x$ at the point $(\frac{\pi}{4}, 1+\sqrt{2})$.

[4] Use Newton's Method to approximate a root of each of the following. Continue until two successive approximations differ by less than 0.001.

(a) $f(x) = x^3 + x - 1$ start at $x = 1$; (ans: 0.682)

(b) $f(x) = 3\sqrt{x-1} - x$ start at $x = 1.2$ (ans:1.146)

(c) $f(x) = x + \sin(x+1)$ start at your choice (ans: -0.489)

[5] Sketch the graphs of $f(x) = x^2$ and $g(x) = \cos x$. Use Newton's Method to approximate the points of intersection (within 10^{-3}).

[6] Try Newton's Method in the following cases; all is not well. Explain.

(a) $f(x) = 2x^3 - 6x^2 + 6x - 1$ starting at $x = 1$

(b) $f(x) = -x^3 + 3x^2 - x + 1$ starting at $x = 1$ (run a couple of iterations).

[7] Calculate the following limits:

(a) $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x^2-1}$ (b) $\lim\limits_{x\to \pi/4} \dfrac{\sin x - \cos x}{x - \pi/4}$ (c) $\lim\limits_{x\to\infty} (x - \sqrt{x^2+x})$

[8] Explain why each of the following is an incorrect application of L'Hôpital's Rule.

(a) $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-3} = \lim\limits_{x\to 3} \dfrac{1}{2x} = \dfrac{1}{6}$.

(b) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x+x^2} = \lim\limits_{x\to 0} \dfrac{\sin x}{1+2x} = \lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$.

STRETCHERS

[9] To illustrate the sensitivity of Newton's Method to the initial estimate, apply this method to approximate a root of $f(x) = \sin x - \frac{2}{3}x$ starting at:

(a) $x_0 = 0.904$, (b) $x_0 = 0.905$, (c) $x_0 = 0.906$

[10] There are two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line. Find them. Is this true of any fourth degree polynomial?

Elementary Calculus

Week 3

This week: logarithms, exponentials and graphing. KEYWORDS: exponential function (derivative of), logarithmic function (derivative of), curve sketching

Technique Time

[1] Differentiate the following and state the domain of both f and f':

(a) $f(x) = e^{\sqrt{x}}$ (b) $f(x) = xe^{-x^2}$ (c) $f(x) = (3.7)^x$

(d) $f(x) = e^{\sin x}$ (e) $f(x) = \dfrac{e^{5x}}{1+e^x}$ (f) $f(x) = \dfrac{\ln x}{1+x^2}$

(g) $f(x) = \ln(\sqrt{x} - \sqrt{x-1})$ (h) $f(x) = \ln\left[\left(\frac{2x+1}{x-3}\right)^{8.2}\right]$

[2] Simplify $e^{(\ln x)x}$ then differentiate (a) $y = x^x$, (b) $y = x^{\sqrt{x}}$, (c) $y = (\ln x)^{\ln x}$ (Dec.93)

[3] Find the maximum value of $y = \frac{\ln x}{x}$ for $x > 0$.

[4] Show that the function $f(x) = \ln(1 - e^{-x})$ is strictly increasing for $x > 0$.

[5] Use Newton's Method to determine a value of x such that $x\,e^x = 5$.

[6] Let $f(x) = (x^2 - 3)e^{-x}$. (December 90 Final)

(a) Find the intervals where f is positive and where it is negative.

(b) Calculate the derivative and find the intervals where f is increasing and decreasing.

(c) Find and classify all critical points. Provide the details and justifications.

(d) Find the limits as x goes to $\pm\infty$.

(e) Sketch the graph of $f(x)$.

[7] Same again with:

(a) $f(x) = \frac{x\,e^x}{x-2}$ (b) $f(x) = x^2\,e^{-2x}$ (c) $f(x) = x\,\ln(x)$ (d) $f(x) = \frac{\ln x}{x}$

(e) $f(x) = \frac{x}{x^2-4}$ (f) $f(x) = \frac{x+1}{x^2}\,e^{-x}$ (April 91 Final)

Stretchers

[8] Assume that influenza spreads through the university community at a rate proportional to the product of the number of those infected and the number of those not yet infected. If the total number of students at Carleton is 22000 and $P(t)$ is the number of students infected after t days, express the preceding statement as a differential equation.

[9] Determine the maximum value of the function $f(x) = \dfrac{x+1}{x^4+1}$. Note that you will need Newton's Method here because the derivative has a quartic numerator. (This quartic has a root near 0.5.)

Elementary Calculus — Week 4

Logarithmic differentiation, exponential growth and decay problems, (inverse trigonometric functions).

Test #1: The syllabus is: Differentiation (up to trigs., logs and exponentials), Implicit and logarithmic differentiation, Newton's Method, L'Hopital's Rule and Curve sketching.

Technique Time

[1] Calculate $\frac{dy}{dx}$:

(a) $y = (\sqrt{(}x)^{\ln x}$, for $x > 0$ (b) $y = (\ln x)^{\sqrt{x}}$ for $x > 1$. (c) $x^y = xy$, for $x, y > 0$.

(d) $y = \left(\dfrac{x+1}{(x+2)^4}\right)^{1/3}$ (e) $y = \dfrac{e^x\,(x^3-1)}{\sqrt{2x+1}}$ (f) $y = \sqrt{\dfrac{x^2-1}{x^2-1}}$

[2] Calculate the following limits. If you use L'Hopital's Rule then specify why it applies.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{x\ln x}$ (b) $\displaystyle\lim_{x\to\pi/2} \frac{\cos x}{2+\sin x}$ (c) $\displaystyle\lim_{x\to 0} \frac{2\cos x - (e^x + e^{-x})}{x^2}$

(d) $\displaystyle\lim_{x\to 0} \frac{e^{-x} + \sin x - 1}{x^2}$ (e) $\displaystyle\lim_{x\to 0} (1+2x^2)^{1/x^2}$. (f) $\displaystyle\lim_{x\to\infty} (1+\frac{3}{x})^x$

Exponential G & D: the next three problems ask the three most comon questions: how much will there be at a given time, how much was there at the beginning, when will there be a given amount.

[3] A colony of bacteria is grown under ideal conditions so that the population grows exponentially with time. At the end of 3 hours there are 10^4 bacteria and at the end of 5 hours there are 4×10^4 bacteria. How many were present initially?

[4] The rate of decomposition of radioactive nuclei follows an exponential decay equation $m(t) = c\,e^{kt}$ for some negative constant k. Radium has a *half-life* of 1620 years. This means that if we start with m nuclei now then in 1620 years there will be $m/2$ nuclei and in 3240 years there will be $m/4$ nuclei etc. This determines the constant k.

(a) Show that for radium we have $m(t) = m(0)\left(\frac{1}{2}\right)^{\frac{t}{1620}}$.

(b) How much of a 1 gram sample will remain after 1000 years?

[5] The half-life of Polonium is 140 days. Your sample will not be useful to you after 90% of the nuclei have disintegrated. About how long will your sample be useful.

Stretchers

[6] The largest known prime number in October 1983 was $2^{132049} - 1$. How many decimal digits does it contain? Estimate this prime to three significant digits.

[7] Some times we are interested in the *relative rate of change* $\frac{y'}{y}$ of a function y. For example, if $y(t)$ is the national debt and $y' = \$5 \times 10^6$/day then this is much bigger news if the current national debt is only $\$10^6$ rather than say $\$5 \times 10^9$. Where have you seen relative rates of change y'/y appearing recently?

Elementary Calculus

Week 5

This week we will study inverse trigonometric functions and do some more work on differential equations. KEYWORDS: Function(inverse trigonometric), Inverse trigonometric functions, Exponential growth and decay.

Technique Time

[1] Confirm the following (remember to work in radians):

(a) $\arctan(\sqrt{3}) = 1.0472$ (b) $\cos^{-1}(0.75) = 0.7227$ (c) $\sin^{-1}(-0.123) = -0.123$

(d) $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} = 0.7854$ (e) $\tan^{-1}(2.498) = 1.19$ (f) $\sin^{-1}(2.498) = error.$

[2] For each of the following state the domain of the function and differentiate:

(a) $\arctan(e^x)$ (b) $\sin^{-1}\left(x^3\right)$ (c) $\cos^{-1}\left(\sqrt{x}\right)$ (d) $\ln\left(\tan^{-1} x\right)$

[3] Determine a function $f(x)$ with the derivative:

(a) $f'(x) = \frac{x}{1+x^4}$ (b) $f'(x) = \frac{x^3}{1+x^4}$ (c) $f'(x) = \frac{1}{\sqrt{1-4x^2}}$

[4] Radioactive decay. Virtually all living things take up carbon as they grow. This carbon comes in two principal forms: normal stable C^{12} and radioactive carbon C^{14} . The C^{14} decays into C^{12} with a half-life of 5730 years. While a tree is alive the proportion of C^{14} will be fixed at the atmospheric ratio, but once the tree dies the C^{14} decays and the ratio changes.

Suppose that an ancient piece of charcoal contains only 15% as much of the radioactive isotope as a piece of modern charcoal. How long ago was the ancient tree burned? [ANS: 15,683 years]

[5] One model of the evolution of the English language specifies that 77% of all words disappear (or are replaced) ever 1000 years. According to this model, what percentage of the English vocabulary available to Chaucer in A.D. 1400 is still in use today?

[6] Solve the following differential equations.

(a) $\frac{dy}{dx} = 2x - 7$ (b) $\frac{dy}{dx} = x\sqrt{x^2+1}$ (c) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y > 0$ (d) $\frac{dy}{dx} = \frac{x}{y}$, $y > 0$

[7] Determine the values of the constants k and w such that

(a) $y = \frac{k}{x} + w$ is a solution of the DE $\frac{dy}{dx} = \frac{23-y}{x}$

(b) $y = w\, e^{k\,x^2}$ is a solution of the DE $y' = xy$

Stretchers

[8] Suppose that a piece of bone is examined in the year 2000 and the C^{14} content, as a percent of the original amount, is measured as a value R. Estimate the bone's age if (a) $R = 16\%$, (b) $R = 17\%$, (c) $R = 18\%$.

Elementary Calculus **Week 6**

This week we will begin work on integration. We will study the Definite Integral and the Fundamental Theorem of Calculus. KEYWORDS: Definite integral, Integral (definite), Fundamental Theorem of Calculus

Technique Time

[1] The following table gives readings of the flow rate in a pipe measured at 1 minute time intervals. Estimate the total flow during the five minute period. Assuming that the flow is monotonically decreasing, give both an upper and a lower estimate.

time (min)	0	1	2	3	4	5
flow rate (l/min)	50	31	19	12	8	7

[2] A car comes to a complete stop 5 seconds after the driver slams on the breaks. While the brakes are on the speed was recorded as in the table. Estimate the total distance travelled by the car in those 5 seconds.

time (sec)	0	1	2	3	4	5
speed (m/sec)	29	20	13	8	3	0

[3] Suppose that $\int_a^b f(t)\,dt = 12$ and $\int_a^b g(t)\,dt = -2$. What is the value of $\int_a^b 2f(t) - 3g(t)\,dt$?

[4] Write each of the following sums as a single definite integral $\int_a^b f(x)\,dx$.

(a) $\int_3^{12} f(x)\,dx + \int_0^3 f(x)\,dx + \int_{12}^{15} f(x)\,dx$ (b) $\int_3^{12} f(x)\,dx - \int_{15}^{12} f(x)\,dx$

(c) $\int_3^{12} f(x)\,dx + \int_{12}^{20} f(x)\,dx - \int_3^{6} f(x)\,dx$

[5] Determine an antiderivative $F(x)$ of $f(x) = \dfrac{x}{\sqrt{1-x^4}}$ and hence evaluate $\int_0^{1/\sqrt{2}} \dfrac{x}{\sqrt{1-x^4}}$.

[6] Find the derivatives of the following functions of x:

(a) $\int_0^x \sqrt{\sin t}\,dt$ (b) $\int_{23}^x \sqrt{\sin t}\,dt$ (c) $\int_{x^2}^0 \sqrt{\sin t}\,dt$ (d) $\int_0^{x^3} \dfrac{1}{1+t^6}\,dt$

(Parts (c) and (d) are from the April 1991 and the August 1990 Final examinations.)

[7] Solve the following differential equations.

(a) $\frac{dy}{dx} = 2x - 7$ (b) $\frac{dy}{dx} = x\sqrt{x^2+1}$ (c) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y > 0$ (d) $\frac{dy}{dx} = \frac{x}{y}$, $y > 0$

[8] Determine the values of the constants k and w such that

(a) $y = \frac{k}{x} + w$ is a solution of the DE $\frac{dy}{dx} = \frac{23-y}{x}$

(b) $y = w\,e^{k\,x^2}$ is a solution of the DE $y' = xy$

Elementary Calculus

Week 7

The plan is to consolidate some old topics and look at a couple of new ideas. We will do some numerical integration, some area problems and look at the F.T.O.C. We will also start substitution techniques.

[1] Use the average or trapezoidal method to approximate $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$ using 4 equal subintervals.

[2] Use an antiderivative to calculate $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$ exactly.

[3] Calculate the area under the curve $y = \sin x$ and above the x-axis between $x = 0$ and $x = \pi$.

[4] Give an antiderivative for each of the following:

(a) $\cos x$ (b) $\sqrt{x}$ (c) $\frac{1}{\sqrt{x}}$ (d) $x^3 + \sin x$ (e) $e^{x^2}(2x)$ (f) $\frac{\sin(\sqrt{x})}{2\sqrt{x}}$

[5] Evaluate the following integrals:

(a) $\displaystyle\int_4^{25} \frac{1}{\sqrt{x}}\,dx$ (b) $\displaystyle\int_{\pi/4}^{\pi} x^3 + \sin x\,dx$ (c) $\displaystyle\int_3^{15} e^{x^2}(2x)\,dx$ (d) $\displaystyle\int_2^7 \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx$

[6]

(a) $\displaystyle\int \frac{dx}{\sqrt{3x+5}}$ (b) $\displaystyle\int_1^7 \frac{dx}{\sqrt{3x+5}}$ (c) $\displaystyle\int_7^1 \frac{dx}{\sqrt{3x+5}}$ (d) $\displaystyle\int \frac{\sec^2 x}{\sqrt{3\tan x+5}}\,dx$

[7] (a) Let $F(x) = \displaystyle\int_0^x 1 + t^2\,dt$. Calculate $\frac{d\,F(x)}{d\,x}$.

(b) Calculate $\frac{d\,G(x)}{d\,x}$ for $G(x) = \displaystyle\int_0^{\sin x} \frac{-t}{\sqrt{1-t^2}}\,dt$.

[8] And as a review, solve for the values of A and k given that

$$A\,e^{20k} = 300$$

$$A\,e^{40k} = 500$$

[9] Differentiate $g(x) = x\,\ln(\tan^{-1}(x))$. What is the domain of this function? (April 1995)

[10] An initial amount of 10 grams of cobalt is subject to radioactive decay. The amount at time t (measured in grams) is denoted by $y(t)$. The function $y(t)$ satisfies the differential equation $y'(t) = k\,y(t)$. Radioactive cobalt has a half life of 5.3 years. That is, after 5.3 years, only half of the original amount remains.

(a) Find the constant k.

(b) Find the weight of the amount of cobalt after 2 years.

Elementary Calculus

Week 8

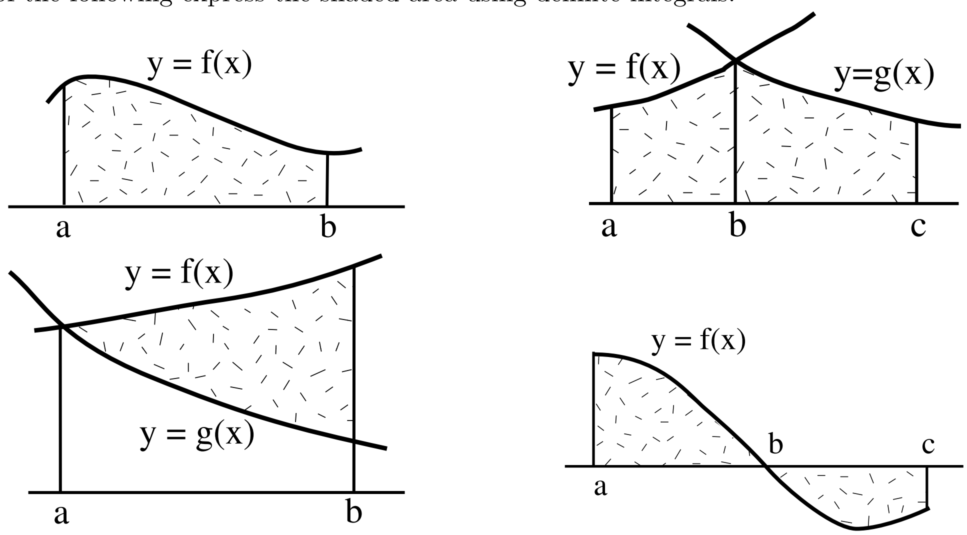
Substitution Integrals and Area Problems. KEYWORDS: Integration(by substitution), Area(under a curve, between two curves)

[1] Held over from last week: *the FTOC questions*.

(a) Let $F(x) = \int_0^x 1 + t^2 \, dt$. Calculate $\frac{d\,F(x)}{dx}$.

(b) Calculate $\frac{d\,G(x)}{dx}$ for $G(x) = \int_0^{\sin\,x} \frac{-t}{\sqrt{1-t^2}} \, dt$.

[2] In each of the following express the shaded area using definite integrals.



[3] Sketch the specified region and calculate its area:

(a) bounded by the curves $y = e^{x/2}$ and $y = \frac{1}{x^2}$ and the lines $x = 2$ and $x = 3$;

(b) bounded by the curves $y = x$ and $y = \frac{4}{x}$ and the lines $y = 0$, $x = 1$ and $x = 4$;

(c) bounded by the curves $y = x^2$ and $y = x + 2$;

(d) bounded by the curves $y = x$ and $y = \frac{1}{x^2}$ and the line $y = 2$;

(e) bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

[4] Substitution integrals.

(a) $\displaystyle\int e^{x^2}\, x\, dx$ (b) $\displaystyle\int (e^x + x)^2\,(e^x + 1)\, dx$ (c) $\displaystyle\int \frac{\sin\,x \cos\,x}{\sqrt{1 + 3\sin^2\,x}}\, dx$

(d) $\displaystyle\int_3^7 e^{x^2}\, x\, dx$ (e) $\displaystyle\int_0^{\ln\,5} (e^x + x)^2\,(e^x + 1)\, dx$ (f) $\displaystyle\int_0^{\pi/2} \frac{\sin\,x \cos\,x}{\sqrt{1 + 3\sin^2\,x}}\, dx$

[5] More substitution integrals — each of these illustrates an important point:

(a) $\displaystyle\int \frac{1}{4x^2 + 8x + 13}\, dx$ (b) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$ (c) $\displaystyle\int \cos^3\, 4x\, dx$ (d) $\displaystyle\int \tan\,x\, dx$

(e) $\displaystyle\int \tan^3\, dx$ (remember that $\tan^2\,x = \sec^2\,x - 1$) (f) $\displaystyle\int_4^9 \frac{\ln(2\sqrt{x})}{\sqrt{x}}\, dx$ (Final Exam '91)

(g) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{2\sqrt{x}\,(x + 1)}\, dx$ (h) $\displaystyle\int \frac{x}{\sqrt{2x + 5}}\, dx$ (i) $\displaystyle\int (x^2 + x + 1)\sqrt{x + 1}\, dx$

Elementary Calculus

Week 9

This week we will look at more integration methods and calculate some volumes. Assignment #3 is due on Friday. KEYWORDS: Integration(by parts), Completing the square, Volume(of a solid of revolution)

[1]

(i) $\displaystyle\int \frac{1}{x^2+2}\,dx$ (ii) $\displaystyle\int \frac{1}{4x^2-4x+3}\,dx$ (iii) $\displaystyle\int \frac{1}{x^2+6x+14}\,dx$

[2]

(i) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$ (ii) $\displaystyle\int \frac{1}{\sqrt{3+6x-9x^2}}\,dx$ (iii) $\displaystyle\int \frac{\sec^2 x}{\sqrt{4-\tan^2 x}}\,dx$

[3] Suppose that a circular cone sits with its apex at the origin and its axis along the x-axis from $x=0$ to $x=10$. Suppose that the (vertical) slice at a given value of x has area $A(x)=3x^2$. What is the volume of the cone?

[4] Calculate the volume of the solid obtained when the specified region is revolved about the given axis:

(a) bounded by $y=x^2, x=1, y=0$, around the x-axis;

(b) bounded by $y=x^2+1, y=3-x^2$, about the x-axis;

(c) bounded by $y=x^2, y=4, x=0, x=2$, about the y-axis.

[5] Sketch the region bounded by the curves $y=3x-x^2$ and $y=x$. Calculate the volume of the solid generated by revolving this region around the y-axis.(Dec. 90 Exam.)

[6] Integration by parts.

(a) $\displaystyle\int x\sin x\,dx$ (b) $\displaystyle\int x\,e^x\,dx$ (c) $\displaystyle\int_0^1 x^2\,e^x\,dx$ (d) $\displaystyle\int_0^1 x^2\,\sin^{-1} x\,dx$

(e) $\displaystyle\int_1^{4e} \ln x\,dx$ (f) $\displaystyle\int e^x\cos x\,dx$ (g) $\displaystyle\int x(\ln x)^2\,dx$ (h) $\displaystyle\int \tan^{-1} x\,dx$

[7] You choose the method.

(a) $\displaystyle\int (\sec^2 x)\,2^{\tan x}\,dx$ (b) $\displaystyle\int \frac{1}{4x^2+8x+13}\,dx$ (c) $\displaystyle\int x\sec^2 2x\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{1-x}}\frac{1}{\sqrt{x}}\,dx$

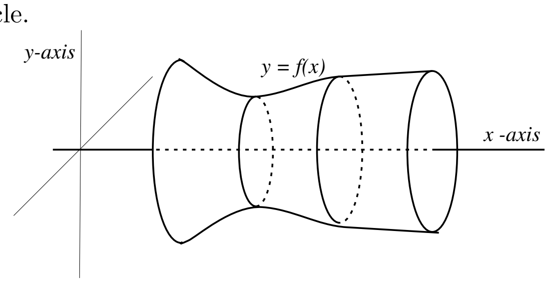
(e) $\displaystyle\int \frac{\sin 2x}{3+\cos 2x}\,dx$ (f) $\displaystyle\int_0^{13} x(2x+1)^{1/3},dx$ (g) $\displaystyle\int_{-\ln 2}^{-(\ln 2)/2} \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$ (h) $\displaystyle\int (x^2+x+1)\,e^x\,dx$

Week 10

This week we will calculate some volumes and look at some more integration techniques.

[1] Suppose that a circular cone sits with its apex at the origin and its axis along the x-axis from $x = 0$ to $x = 10$. Suppose that the (vertical) slice at a given value of x has area $A(x) = 3x^2$. What is the volume of the cone?

[2] Suppose that a solid object is sitting along the x-axis from $x = a$ to $x = b$ and that for every x the slice through the object is a circle.



If the radius at x is $f(x)$ then the area of the slice there is _______ and the total volume is _____________.

[3] Calculate the volume of the solid obtained when the specified region is revolved about the given axis:

(a) bounded by $y = x^2, x = 1, y = 0$, around the x-axis;

(b) bounded by $y = x^2 + 1, y = 3 - x^2$, about the x-axis;

(c) bounded by $y = x^2, y = 4, x = 0, x = 2$, about the y-axis.

[4] Sketch the region bounded by the curves $y = 3x - x^2$ and $y = x$. Calculate the volume of the solid generated by revolving this region around the y-axis.(Dec. 90 Exam.)

Some Trigonometry Facts

$\sin^2 x + \cos^2 x = 1$	$\sin 2x = 2(\sin x)(\cos x)$	$\cos 2x = \cos^2 x - \sin^2 x$
$sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + cos\, 2x)$	$\tan^x +1 = \sec^2 x$

[5]

(a) $\int \sin^2 x\, dx$ (b) $\int \sin^3 x\, dx$ (c) $\int \sin^4 x\, dx$ (d) $\int \tan^2 x\, dx$

(e) $\int x \sin x \cos x\, dx$ (f) $\int \sin x \cos^2 x\, dx$ (g) $\int \sin^9 x \sin 2x\, dx$

[6] A different sort of trig. substitution. For the first one let $x = \sin u$.

(a) $\int \sqrt{1 - x^2}\, dx$ (b) $\int \sqrt{49 - 9x^2}\, dx$

Week 11 and 12

This week our new topics are integration by partial fractions and using integrals to solve differential equations. We will also do more volumes for practice. This handout covers two weeks; it is the last Weekly page.

[1] Partial Fractions: Solve for A, B, C in each of the following.

(a) $\dfrac{x}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

(b) $\dfrac{x^2+1}{(x+1)^2\,(x-3)} = \dfrac{A}{(x+1)^2} + \dfrac{B}{x+1} + \dfrac{C}{x-3}$

(c) $\dfrac{x^2}{x^2-4} = A + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

(d) $\dfrac{x^2+x+1}{(x^2+x+2)(x-2)} = \dfrac{Ax+B}{x^2+x+2} + \dfrac{C}{x-2}$

[2]

(a) $\displaystyle\int \frac{x}{(x-3)(x+2)}\,dx$

(b) $\displaystyle\int \frac{x^2+1}{(x+1)^2\,(x-3)}\,dx$

(c) $\displaystyle\int \frac{x^2}{x^2-4}\,dx$

[3]

(a) $\displaystyle\int_4^{10} \frac{10t}{(t+2)(t-3)}\,dt$

(b) $\dfrac{2x-4}{(x+1)^2\,(x-3)}\,dx$

(c) $\dfrac{-7x-23}{(x-6)(x^2+4x+5)}\,dx$

[4] $\displaystyle\int \frac{14x^5+37x^4-16x^3-2x^2+3x+1}{2x^3+5x^2-3x}\,dx$

[5] Compare the two integrals $\displaystyle\int \frac{1}{x^2-4}\,dx$ and $\displaystyle\int \frac{1}{x^2+4}\,dx$.

[6] Find a solution for each of the following differential equations that includes the given point.

(a) $\dfrac{dy}{dx} = y\sec^2 x$; $(x,y) = (0,42)$

(b) $\dfrac{dy}{dx} = \dfrac{\sin x}{y}$; $(x,y) = (0,1)$

(c) $\dfrac{dy}{dx} = \dfrac{y}{\sqrt{1-x^2}}$; $(x,y) = (0,7)$

(d) $(x+7)\dfrac{dy}{dx} = y^2+1$; $(x,y) = (2,1)$

[7] Sketch the triangular region bounded by the lines $y = x+1$, $y = 5$ and $x = 2$. Calculate the volume of the solid obtained by revolving this region about the x-axis.

[8] Sketch the following regions and calculate the volume of the solid obtained by revolving about the x-axis.

(a) Bounded by $y = mx$, $y = 0$ and $x = h$ for constants $m, h > 0$.

(b) Bounded by $y = s$, $y = r$, $x = a$ and $x = b$ for constants s, r, a, b with $r > s > 0$ and $b > a$.

(c) Bounded by $y = 0$ and $y = \sqrt{R^2 - x^2}$ for some constant $R > 0$.

[9] Calculate the area of the circle $x^2 + y^2 = R^2$.

[10] Calculate the volume of the cylinder with height h and radius r.

Stretchers

[11] Sketch the circle with equation $x^2 + (y-h)^2 = r^2$ with h and r constants $h > r > 0$. Revolve this circle around the x-axis and calculate the volume of the resulting doughnut (also known as a *torus* to the cognoscente).