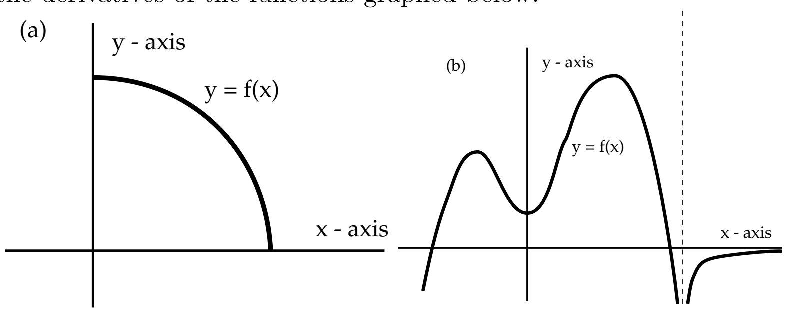
Elementary Calculus

Tutorial 1

Today's tutorial is a review of some ideas of differential calculus. Please work in groups of at most 4 – one set of solutions per group. Note the curious marking scheme. The idea is to encourage you to work carefully on the first few questions and avoid racing to the end.

[1](4 marks) Sketch the derivatives of the functions graphed below.



[2](3 marks) Calculate the derivatives of the following functions:

(a) $f(x) = \sqrt{x + x^3}$ (b) $f(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$

[3](2 marks) Find the equation of the tangent line to:

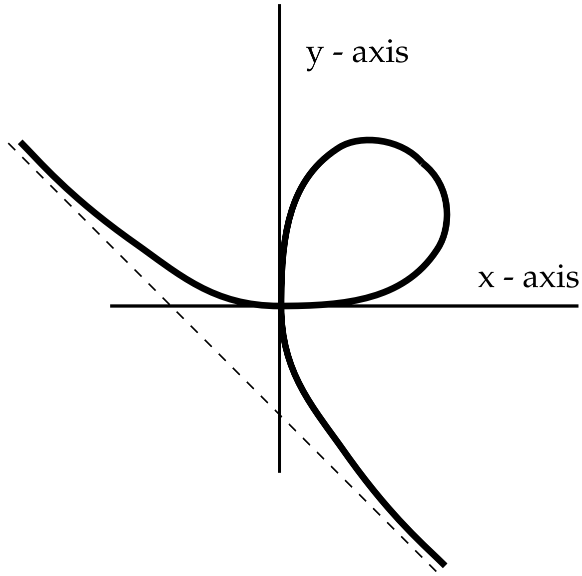
(i) the curve $x^2y + y^2x - xy = 4$ at the point (2,1);

(iii) the curve $\sqrt{xy} = x - 2y$ at the point (4,1).

[4](1 mark) Suppose that $h(x)$ is a function such that $h(0) = 22$ and such that for all x we know that the derivative of h satisfies $2 \leq h'(x) \leq 6$. This is not much to go on but still it tells us something. Give upper and lower bounds on the number $h(10)$. This means that we want numbers L and U such that we are sure that $L \leq h(10) \leq U$.

Stretchers

[5](0 marks) The curve $x^3 + y^3 - 6xy = 0$ is called the *Folium of Descarte*. It looks like this:



(i) Calculate the slope of the tangent line at a general point.

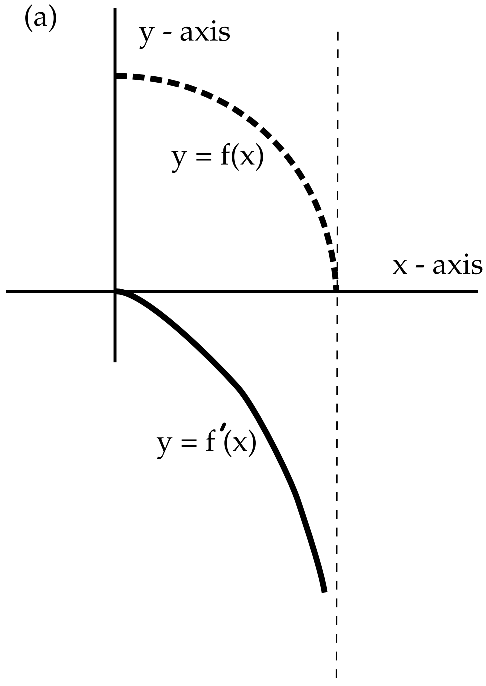
(ii) What happens to your formula at (0,0)?

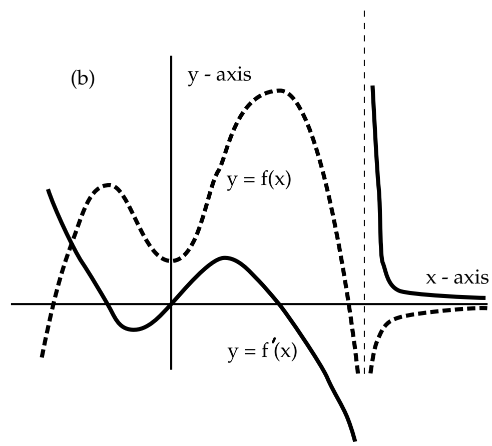
(iii) Explain what goes wrong with implicit differentiation at (0,0).

Answers for Tutorial #1

These are answers and not full solutions.

[1]





[2]

(a) $f'(x) = \dfrac{1+3x^2}{2\sqrt{x+x^3}}$ (b) $f'(x) = \dfrac{-1}{\sqrt{x}}\dfrac{1}{(\sqrt{x}-1)^2}$

[3] (a) The first step in implicit differentiation is

$$2xy + x^2y' + 2yy'x + y^2 - xy' - y = 0.$$

Using $(x, y) = (2, 1)$ this yields $y' = \dfrac{-2}{3}$. The equation of the tangent is $(y - 1) = (-2/3)(x - 2)$ which is $2x + 3y = 7$.

(b) The first step gives $\dfrac{1}{2\sqrt{xy}}\,(y + x\,y') = 1 - 2y'$. Using $(x, y) = (4, 1)$ this implies that $y' = 1/4$. The equation of the tangent is $y - 1 = \dfrac{1}{4}(x - 4)$, that is $4y = x$.

[4] The rate of increase of the function is at least 2 so the total increase from 0 to 10 must be at least 20. So the value of h at 10 is at least $h(0) + 20 = 42$. Similarly the rate of increase of the function is at most 6 at all times from $x = 0$ to $x = 10$ so $h(10) \le h(0) + 60 = 82$. Therefore $42 \le h(10) \le 82$.

[5] (a) The slope of the tangent is $y' = -\dfrac{3x^2 - 6y}{3y^2 - 6x}$.

(b) At (x, y) both numerator and denominator are zero so the formula is undefined at this point.

(c) The problem is that there are two tangent lines at $(0, 0)$ so there is no way for the formula to give both slopes. Even in a small region around $(0, 0)$ the curve is not a function.

Elementary Calculus

Tutorial 2

Today we are working on derivatives of trigonometric functions, graphing and Newton's Method.

This weekend would be a great time to review **logarithms**. Look on page 210 in 3rd edition or page 211 of the 2nd edition or under logarithms (properties) in the index. You should be able to solve equations like $\log(x-5) = 3$ and $55(10)^{4x} = 3662$.

[1](2 marks) Warm up with some derivatives. Differentiate the following functions:

$$\text{(a) } f(x) = \tan\left(\frac{x^2}{\sqrt{x+4}}\right) \qquad \text{(b) } g(\theta) = \frac{\theta}{\sin^2\theta}$$

[2](4 marks) In each of the following three cases, draw a possible graph $y = f(x)$ for which the given data are true. Be sure to label the axes. There are infinitely many different correct answers.

(i) $f(x)$ is defined for all x, $f(x) = 0$ at $x = 0$,

$f'(x) = 0$ at $x = -2$ and 1, $f' > 0$ for $x < -2$ and $x > 1$, $f'(x) < 0$ for $-2 < x < 1$,

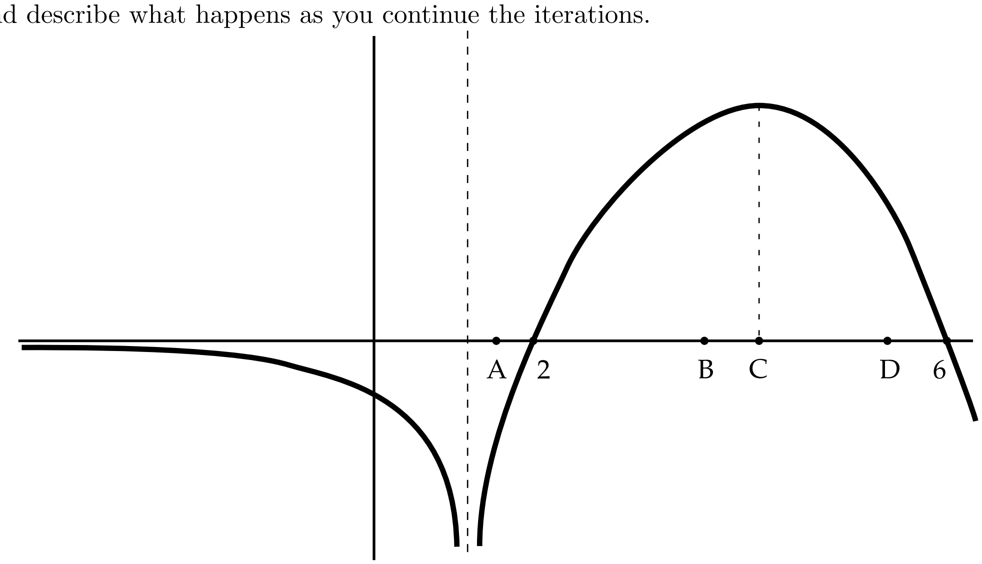
$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.

(ii) $f(x)$ is defined except at $x = -1$ and $x = 3$, $f'(x) > 0$ always,

$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 1$.

(iii) $f(x)$ defined for all x, $f'(x) = 0$ at $x = 1, 2, 3$, $f'(x) > 0$ for $x < 1$, $\lim_{x\to+\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = -\infty$.

[3](3 marks) For each of the points A,B,C,D in the graph below, imagine starting Newton's Method at that point and describe what happens as you continue the iterations.



[4](1 mark) Use Newton's Method to calculate an approximate solution x of the equation $\sin x = 1 - x$. (Try to get the second decimal place correct.)

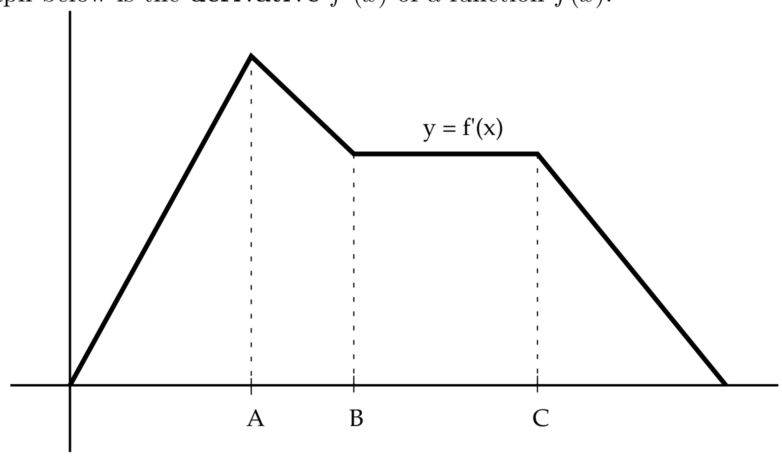
Stretchers

[5](0 marks) Try Newton's Method on the following functions.

(i) $x^2 + 1$ (Note that there are no real roots. Sketch a graph to see what happens.)

(ii) $x^3 - 12x^2 - 16x - 64$ starting at $x_0 = 0$, (Go for at least 4 iterations – something special happens.)

[6](0 marks) The graph below is the **derivative** $f'(x)$ of a function $f(x)$.



(i) Sketch the graph of the function $f(x)$. Does your graph behave properly at the points A, B and C?

(ii) Is there more than one function with this same derivative? Explain.

(iii) Sketch the graph of the second derivative $f''(x)$; in other words, the graph of the derivative of the function graphed above.

[7](0 marks) The graphs of $\sin x$ and $\cos x$ intersect once between 0 and $\pi/2$. Determine the angle between the two curves at the point where they intersect? (You will need to think about how the angle between two curves should be defined.)

Solutions to Tutorial #2

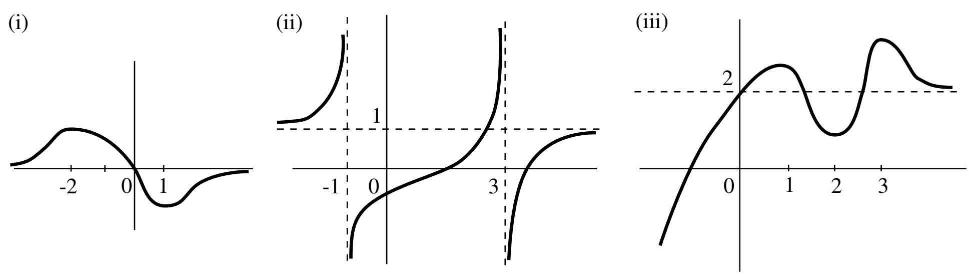
[1] (a) Think of the function as $y = \tan\left(x^2\,(x+4)^{-1/2}\right)$ we differentiate to get

$$\sec^2\left(\frac{x^2}{\sqrt{x+4}}\right)\left[\frac{2x}{\sqrt{x+4}} + \frac{-x^2}{2(x+4)^{3/2}}\right] = \sec^2\left(\frac{x^2}{\sqrt{x+4}}\right)\frac{3x^2+16x}{2(x+4)^{3/2}}$$

(b) Using the quotient rule we get:

$$\frac{d\,y}{d\,\theta} = \frac{\sin^2\,\theta - \theta\,2\,\sin(\theta)\,\cos(\theta)}{\sin^4\,\theta} = \frac{\sin\,\theta - 2\,\theta\,\cos\,\theta}{\sin^3\,\theta}$$

[2] Thee is not a unique correct answer to any of the parts of this question. I give only one graph for each case; yours might look quite different and still be correct.



[3] • From A the iterations will lead to the root 2.

• From B the first iteration produces a negative value for x_1. The succesive iterations will now take us to points farther and farther to the left – away from both roots.

• The derivative is 0 at the point C so even one iteration is impossible since it would require a division by zero.

• From the point D the first iteration will be beyond 5, but following iterations will bring us back to the root 5

[4] Finding an x such that $\sin\,x = 1 - x$ is the same as finding a root of $x + \sin\,x - 1 = 0$. So the function to iterate is

$$x - \frac{x - \sin\,x - 1}{1 + \cos\,x} = \frac{x\,\cos\,x - \sin\,x + 1}{1 + \cos\,x}.$$

Starting at 1 leads to the approximate root 0.510907.

Elementary Calculus

Tutorial 3

The topic today is modelling with differential equations. You may not have seen a problem like this before but you do have all the tools required.

The following paragraph describes a situation that can be modelled by a differential equation.

> Toxic mercury is pouring into a lake from a factory at a constant rate of 2 mg / min. Assume that the other water flowing into the lake is free of mercury and the mercury mixes evenly throughout the lake. Water is flowing into and out of the lake at a constant rate of 1500 l/min. The total constant volume of water in the lake is 8×10^{11} liters.

What we want to know is the mercury concentration in the future – the near future and the distant future. Let $C(t)$ be the concentration at time t. We could accomplish our goal if we had a formula for $C(t)$. The paragraph is giving us some information about the rate of change of the concentration; this leads to a differential equation.

[1](5 marks) (i) What are the units of $C(t)$? How is the mass of mercury in the lake related to its concentration? How is the rate of change of the mass related to the rate of change of the concentration?

(ii) Mercury enters the lake; this will increase the concentration. At what rate? Give a formula and an explanation.

(iii) Mercury leaves the lake; this will decrease the concentration. At what rate? Give a formula and an explanation.

[2](2 marks) What do the answers to (i) and (ii) have to do with the derivative $\frac{dC(t)}{dt}$? Write down a differential equation that models the rate of change of mercury in the lake. Justify your answer. The DE will have the form $\frac{dC(t)}{dt} = a + bC(t)$ for some constants a and b. (You are not expected to solve this DE right now.)

[3](2 marks) By looking at your differential equation determine values of the concentration $C(t)$ such that starting from this concentration:

(i) $C(t)$ will not change with time;

(ii) $C(t)$ will decrease with time;

(iii) $C(t)$ will increase with time.

[4](1 mark) What is the predicted long term mercury concentration in the lake? Why? Does this concentration depend on the initial concentration?

Stretchers

[5](0 marks) The solution of the DE in [2] has the form $C(t) = u\ e^{kt} + c$ for some constants u, k, c. By differentiating $u\ e^{kt} + c$, calculate values for u, k, c and hence a solution of the DE of question [2].

[6](0 marks) Here are five functions, each listed with its derivative. Below are the graphs of the functions in some order. Pair up the functions with their graphs. Then use this set of examples to test your understanding of critical points, intervals of increase and decrease, limits at infinity, vertical asymptotes.

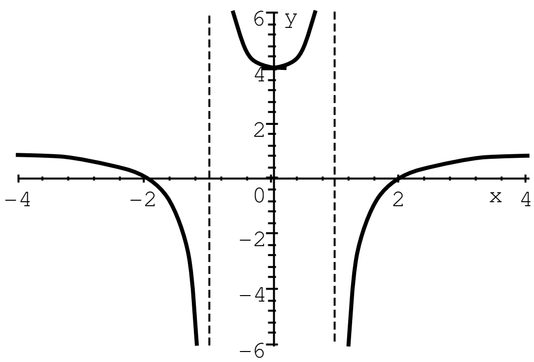
(i) $y = 5 - \dfrac{1}{x^2+1}, \quad y' = \dfrac{2x}{(x^2+1)^2}$

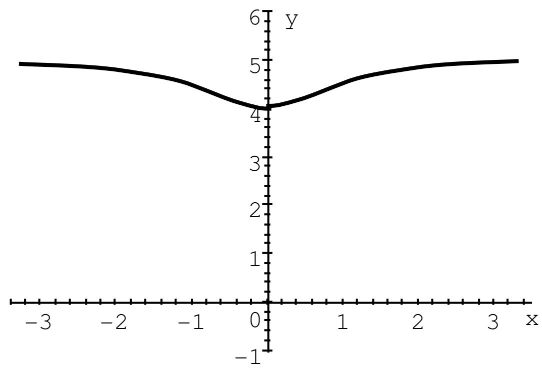
(ii) $y = \dfrac{-6x}{\sqrt{4x^2+5}}, \quad y' = \dfrac{-30}{(4x^2+5)^{3/2}}$

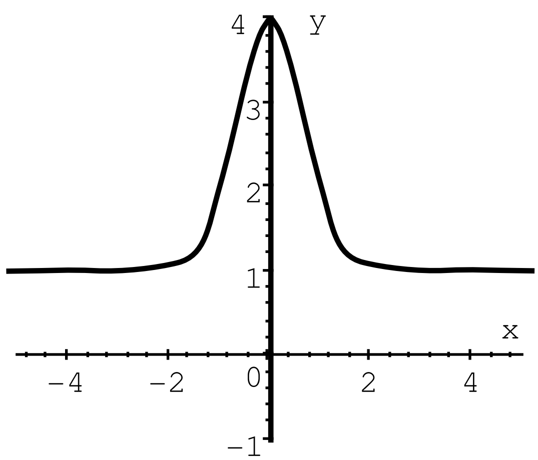
(iii) $y = \dfrac{x^2-4}{x^2-1}, \quad y' = \dfrac{6x}{(x^2-1)^2}$

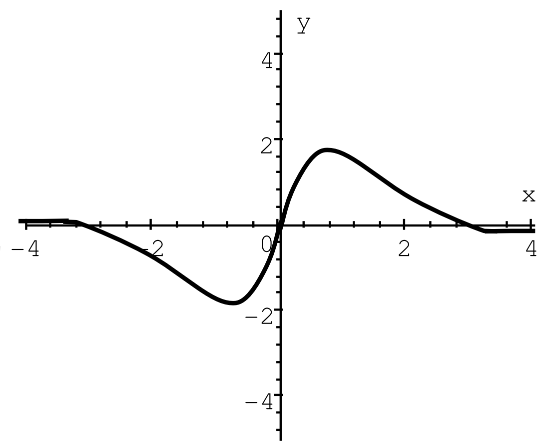
(iv) $y = \dfrac{4\sin(x)}{x^2+1}, \quad y' = 4\dfrac{x^2\cos x + \cos x - 2x\sin x}{(x^2+1)^2}$

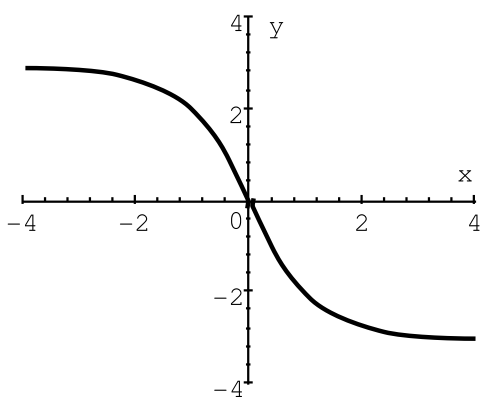
(v) $y = 3\,e^{-x^2}+1, \quad y' = -6\,x\,e^{-x^2}$











Solutions to Tutorial 3

Sections A,B,C,D (Mortimer)

[1] (i) The concentration $C(t)$ is measured in units of mg /litre. The mass M of mercury is equal to the concentration times the volume of the lake $M = C(t)\ 8 \times 10^{11}$. Similarly the rate of change of M is 8×10^{11} times the rate of change of $C(t)$.

(ii) The mercury enters the lake only from the factory. It enters at the rate of 2 mg/min. This increases the concentration at the rate $\dfrac{2}{8 \times 10^{11}}$ mg / l min.

(iii) The mercury leaves the lake in the outflow. The <u>mass</u> flows out at the rate $C(t)\,1500$ mg/min. This decreases the concentration at the rate $\dfrac{C(t)\,1500}{8 \times 10^{11}}$ mg/l min.

[2] The net rate of change of mercury is the answer to (ii) minus the anser to (iii). Thus

$$\frac{d\,C(t)}{d\,t} = \frac{2}{8 \times 10^{11}} - \frac{C(t)\,1500}{8 \times 10^{11}}$$

[3] The concentration will increase if the derivative is positive, will decrease if the derivative is negative and will remain the same if its rate of change is 0,. So we pick three values for $C(t)$ that make $\dfrac{2}{8 \times 10^{11}} - \dfrac{C(t)\,1500}{8 \times 10^{11}}$ zero, positive and negative, respectively.

(i) Take $C(t) = \dfrac{2}{1500}$. This makes $\dfrac{d\,C(t)}{d\,t} = 0$.

(ii) Any value of $C(t)$ smaller than $\dfrac{2}{1500}$ makes the derivative positive.

(iii) Any value of $C(t)$ larger than $\dfrac{2}{1500}$ makes the derivative negative.

The logic here needs a comment. Parts (ii) and (iii) show that if $C(t)$ is not $\dfrac{2}{1500}$ then it will head toward this number. This is what makes $C(t) = \dfrac{2}{1500}$ the stable value

[4] The long term concentration will be $\dfrac{2}{1500}$ mg/l and this does not depend on the initial concentration. At any lower concentration, $C(t)$ is increasing and at any higher concentration $C(t)$ is decreasing.

Tutorial 4

Today there is a word problem, then some inverse trig. derivatives and finally a peak ahead at next week.

[1](4 marks) Suppose that sodium pentobarbitol will anesthetize a dog when its bloodstream contains at least 45 mg of sodium pentobarbitol per kilogram of body weight. Suppose also that sodium pentobarbitol is eliminated exponentially from the dog's blood stream, with a half life of 5 h. What single dose should be administered to anaesthetize a 50 kg dog for 1 h ?

[2](2 marks) Take a break; remember to breathe. Calculate the derivatives of the functions

(a) $y = \sin^{-1}(\sqrt{x})$ (b) $y = \arctan(\sqrt{x})$

[3](3 marks) Let $f(x) = \sin^{-1}\left(\dfrac{x-b}{a}\right)$ where a and b are constants and $a > 0$.

(a) What is the domain of $f(x)$?

(b) Show that the derivative of $f(x)$ is $\dfrac{1}{\sqrt{a^2-(x-b)^2}}$.

(c) Determine a and b such that $5 + 4x - x^2 = a^2 - (x-b)^2$ (say, by completing the square).

(d) Hence solve the differential equation $\dfrac{d\,f(x)}{dx} = \dfrac{1}{\sqrt{5+4x-x^2}}$.

[4](1 mark) Suppose that water is flowing down a pipe and we can measure the rate of flow at any time t. The table below gives readings (in l/min) taken at hourly intervals one afternoon.

time =	1:00	2:00	3:00	4:00	5:00	6:00
flow =	85	76	70	62	40	18

Estimate the total volume of water that passed through the pipe between 1:00 PM and 6:00 PM. Justify your procedure.

Stretchers

[5](0 marks) Use trigonometric identities to get algebraic expressions for:

(a) $\cos(\sin^{-1} x)$ (b) $\cos(\sin^{-1} 2x)$ (c) $\cos(2\sin^{-1} x)$ (d) $\sec^2(\tan^{-1} x)$

Solutions to Tutorial 4

[1] Since the dog weighs 50 kg, we need at least $45 \times 50 = 2250$ mg of sodium pentaborbitol in the blood at the end of the hour. We want an initial mass that will decay, with a half life of 5 hours, into 2250 mg in one hour. The mass at time t is given by

$$m(t) = m(0)\left(\frac{1}{2}\right)^{t/5}$$

and we want $m(0)$ so that $m(1) = 2250$. So we solve $m(0)\left(\frac{1}{2}\right)^{1/5} = 2250$ for $m(0)$. We get $m(0) = 2585$ mg.

[2] (a) For $y = \sin^{-1}(\sqrt{x})$ we have $y' = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\,\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{1-x}\sqrt{x}}$.

(b) For $y = \arctan(\sqrt{x})$ we have $y' = \dfrac{1}{1+(\sqrt{x})^2}\,\dfrac{1}{2\sqrt{x}} = \dfrac{1}{1+x}\,\dfrac{1}{2\sqrt{x}}$.

[3] (a) To have $f(x) = \sin^{-1}\left(\dfrac{x-b}{a}\right)$ defined we need to have $-1 \le \left(\dfrac{x-b}{a}\right) \le +1$. This is the same as $b-a \le x \le b+a$. So the domain of $f(x)$ is all x such that $b-a \le x \le b+a$.

(b) $f'(x) = \dfrac{1}{\sqrt{1-\left(\frac{x-b}{a}\right)^2}}\left(\dfrac{x-b}{a}\right)' = \dfrac{1}{a\sqrt{1-\frac{(x-b)^2}{a^2}}} = \dfrac{1}{\sqrt{a^2-(x-b)^2}}$.

(c) Well now, $5+4x-x^2 = 9-4+4x-x^2 = 3^2-(x-2)^2$. So $a=3$ and $b=2$ will work.

(d) Putting (b) and (c) together, the derivative of $f(x) = \sin^{-1}\left(\dfrac{x-2}{3}\right)$ is $\dfrac{1}{\sqrt{5+4x-x^2}}$. Hence this $f(x)$ is a solution of the differential equation. We can actually also add any constant to this function to get another solution.

[4] There are several ways to estimate the total flow from these data. One way is to suppose that the flow rate was steadily decreasing for the whole time. So the total flow in the first hour is at most 85 l/hr $\times$ 1 hr $= 85$ litres. In the second hour the flow is at most 76 litres. Continuing, the total flow over all five hours is at most $85+76+70+62+40 = 333$ litres.

If the flow is steadily decreasing then in the first hour the total flow is at least 76 litres, in the second hour at least 70 litres and so on. This would give us at least $76+70+62+40+18 = 266$ litres.

We could get fancy and take the average flow rate in each hour. This would give us a total flow of $80.5+73+66+51+29 = 299.5$ litres.

Tutorial 5

Numerical Integration — and as you might guess, the work today is numerical.

If you finish early, Question [4] is rather amusing.

IF YOU NEED GRAPHING PRACTICE then find time to work on [5] on the flip side.

[1](4 marks) Back to the water in the pipe. Here are some flow rates:

time (min)	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(t)$ = flow (L/min)	10	14	15	11	8	7	10

We want to estimate the total flow. As you can see, the flow rate is going up and down. Taking an average on each interval is a reasonable approach.

(a) Calculate the average flow rates on each of the six intervals, that is, the average of the rate at the start of the interval and the rate at the end of the interval.

(b) Hence estimate the total flow $\int_0^3 f(t)\,dt$. Remember that the intervals here have length $\frac{1}{2}$.

[2](3 marks) The same idea estimates any definite integral.

(a) Complete the table below and use it to estimate $\int_0^{10} \sqrt{1+t}\,dt$ by this *average* method.

$t =$	0	2	4	6	8	10
$\sqrt{1+t} =$						

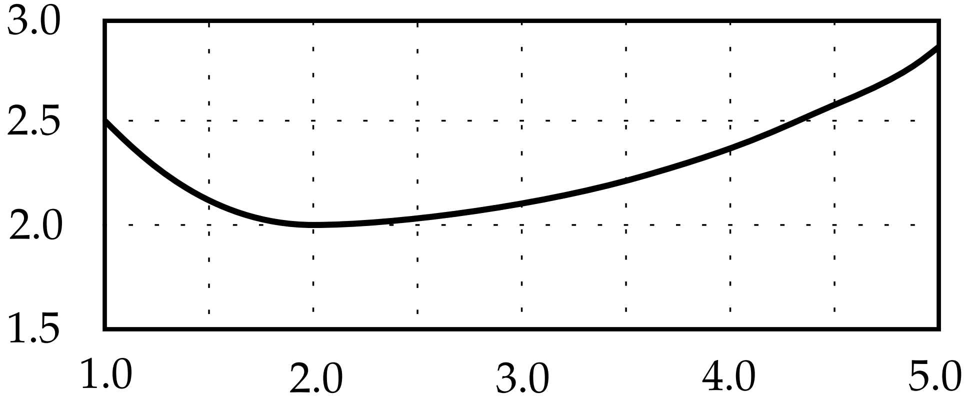
(Did you get the interval length correct?)

(b) If we set $F(t) = \frac{2}{3}(1+t)^{3/2}$ then $F'(t) = \sqrt{1+t}$. Remember our work in class and use $F(t)$ to calculate an exact value for $\int_0^{10} \sqrt{1+t}\,dt$.

[3](3 marks) Now use the average method to estimate $\int_0^3 e^{-x^2}\,dx$ using three intervals. This is an important integral in applications and it can only be calculated numerically; there is no elementary antiderivative.

Stretchers

[4](0 marks) Consider the function f graphed below.



The function $y = f(x)$ is a solution of one of the following differential equations. Decide which one and explain why the others are not possible.

(i) $y' = \dfrac{y-x}{x}$ (ii) $y' = \dfrac{x-y}{x}$ (iii) $y' = \dfrac{x^2-y}{x}$

[5](0 marks) More graphing practice. Suppose that we are graphing a function $y = f(x)$ that is defined for all real $x \neq 3$. Assume that the derivative is

$$f'(x) = \frac{(x-1)^2(x+2)}{(x-3)}.$$

We won't need $f(x)$ itself for this exercise.

(i) Determine the intervals where $f'(x)$ is positive and where it is negative.

(ii) Define, in a sentence, what is meant by a critical point of a function.

(iii) Determine the critical points of $f(x)$.

(iv) Classify each critical point as a local maximum, local minimum or neither. Justify your answers. (For this use the First Derivative Test with the data from (i).)

(v) Sketch a graph that exhibits the behaviour determined in (i),(iii) & (iv).

Solutions for Tutorial 5

[1] (a) The average flow rates are 12, 14.5, 13, 9.5, 7.5 and 8.5 L/min.

(b) Each interval has length $\frac{1}{2}$ min. Thus the total flow is approximately

$$12 \times \frac{1}{2} + 14.5 \times \frac{1}{2} + 13 \times \frac{1}{2} + 9.5 \times \frac{1}{2} + 7.5 \times \frac{1}{2} + 8.5 \times \frac{1}{2} = 32.5.$$

[2] (a) The table of values is

$t =$	0	2	4	6	8	10
$\sqrt{1+t} =$	1	1.73	2.23	2.64	3	3.32

The averages are 1.365, 1.984, 2.441, 2.823, 3.158. The intervals have length 2. Our estimate is

$$1.365 \times 2 + 1.98 \times 2 + 2.441 \times 2 + 2.823 \times 2 + 3.158 \times 2 = 23.54$$

(b) The exact value of the integral is $F(10) - F(0)$ where $F(t) = \frac{2}{3}(1+t)^{3/2}$. So the exact value is $\frac{2}{3}(11)^{3/2} - \frac{2}{3} = 23.655$.

[3]

$x =$	0	1	2	3
$e^{-x^2} =$	1	0.368	0.018	0.0001

The average values are 0.684, 0.193, 0.0090. The intervals have length 1. So the estimate is

$$\int_0^3 e^{-x^2}\,dx \approx 0.684 \times 1 + 0.193 \times 1 + 0.009 \times 1 = 0.886.$$

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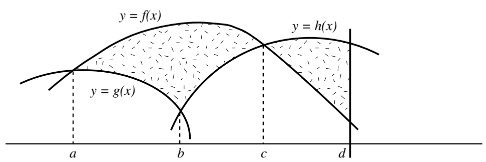
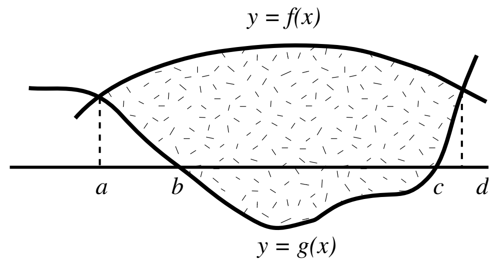
Elementary Calculus

Tutorial 6

Today is mostly drill and practice.

[1](3 marks) Below are two pictures each showing a region of the plane bounded by certain functions. In each case write a formula for the area of the shaded region using definite integrals.





[2](3 marks) (a) Sketch the region bounded by the following curves:

$$y = x^2 - 1, \qquad y = 6/x, \qquad y = 0, \qquad x = 1, \qquad \text{and } x = 6.$$

(b) Calculate the area of this region.

[3](4 marks) Substitution Integrals. Calculate the integrals below. In the first two cases, I have suggested a substitution to simplify the integral. After that it is up to you to pick $u(x)$.

(i) $\displaystyle\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}\,dx$, $u = 1+\sqrt{x}$ (ii) $\displaystyle\int_1^2 \frac{1+\ln x}{x}\,dx$, $u = 1 + \ln x$.

(iii) $\displaystyle\int (\cos^3 x)\sin x\,dx$ (iv) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

[4](0 marks) Here are some more for practice.

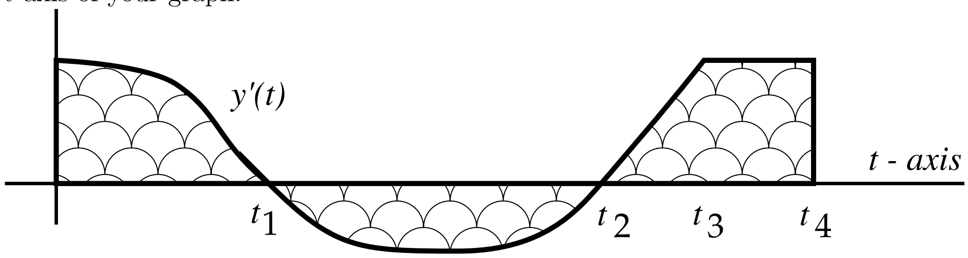
(i) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ (ii) $\displaystyle\int \frac{\sec^2(\sqrt{x+1})}{\sqrt{x+1}}\,dx$ (iii) $\displaystyle\int \sqrt{2x} - \frac{1}{\sqrt{3x}}\,dx$

(iv) $\displaystyle\int \frac{2t+1}{\sqrt{t^2+t}}\,dt$ (v) $\displaystyle\int \tan^2 x\ \sec^2 x\,dx$[Try:$u = \tan x$] (vi) $\displaystyle\int \frac{\cos 3x}{1+\sin 3x}\,dx$

(vii) $\displaystyle\int t\sqrt{t+1}\,dt$ [Trick: let $u = \sqrt{t+1}$ so $u^2 = t+1$ and $2u\,du = dt$.]

Stretchers

[5](0 marks) Back to an old favourite! The graph below shows $y'(t)$ as a function of t. Suppose that each of the three shaded regions has area 2 and that $y = 0$ at $t = 0$. Your job is to draw the graph of $y(t)$. Of course, $y(t)$ is the antiderivative of $y'(t)$ so your knowledge of integration will help. But also $y'(t)$ is the derivative of $y(t)$ so your knowledge of derivatives will help as well. Take care with all the special features that the graph might have (known heights, maxima, minima etc.). Mark the points t_1, t_2, t_3 and t_4 on the t-axis of your graph.



[6](0 marks) Suppose that there is an outbreak of a nonfatal flu in Ottawa. The Medical Officer of Health monitors the rate of change f(t) of the infected population (in new cases/day) and wants to estimate the total number of new cases to expect before the end of the year. The rate of new cases is declining rapidly: there were 1000 new cases reported on Dec. 1 but only 120 new cases on Dec. 10. The MOH decides to use the function $f(t) = \dfrac{a}{t+b}$ to model the rate of change of the infected population.

(i) Taking $t = 0$ on Dec. 1 we have $f(0) = 1000$ and $f(10) = 120$. Calculate the values of a and b in the model.

(ii) Use the model to estimate the total number of new cases to appear between $t = 0$ and $t = 31$ (Dec. 31).

[7](0 marks) The curves $y = \sin x$ and $y = \cos x$ intersect infinitely often. Calculate the area enclosed by these two curves between two succesive intersections.

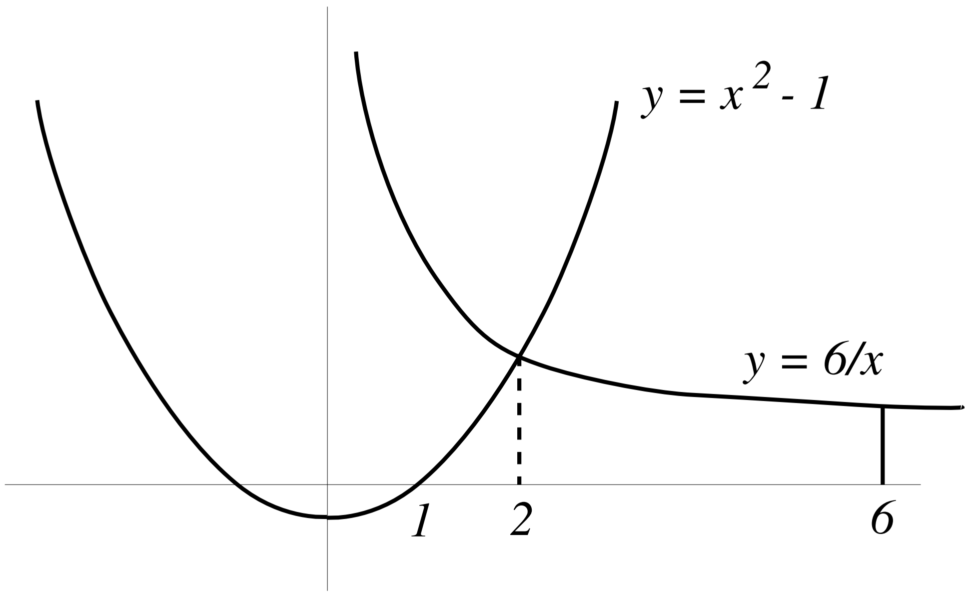
Solutions for Tutorial # 6

[1] We use the formula: area $= \int_a^b TOP - BOTTOM\,dx$. The first area can be written

$$\int_a^b (f(x) - g(x))\,dx + \int_b^c (f(x) - h(x))\,dx + \int_c^d (h(x) - g(x))\,dx.$$

The second area is $\int_a^d f(x) - g(x)\,dx$.

[2] The region looks like this



The intersection of the two curves is at $x = 2$. The area is

$$\int_1^2 x^2 - 1\,dx + \int_2^6 \frac{6}{x}\,dx = \left(\frac{1}{3}x^3 - x\right)\Big|_1^2 + 6\ \ln\ x\Big|_2^6 = \frac{4}{3} + 6\ln 3.$$

Elementary Calculus

Tutorial 7

Today we will practice integration techniques. These calculations are not simple one idea problems. Solving an integral typically involves several steps, a combination of methods, some algebraic finesse, perseverance and confidence. Are you ready? Here we go...and don't worry if you don't get to Question [5]

[1](0 marks) Start off today by checking the following integrals. (How do you check an integral?) These will repay (with interest) a few moments of careful study.

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + C$$
$$\int \frac{1}{a^2+b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$
$$\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$
$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + C.$$

[2](4 marks) Now here are two that are done by parts. (Think carefully about that <u>definite</u> integral.)

(a) $\int \tan^{-1} x\,dx$ (b) $\int_4^{25} \sqrt{x}\ln x\,dx$

[3](4 marks) Some integrals have to be split into two integrals:

(a) Calculate $\int \frac{4x+5}{x^2+9}\,dx$. Write $\int \frac{4x+5}{x^2+9}\,dx = 2\int \frac{2x}{x^2+9}\,dx + 5\int \frac{1}{x^2+9}\,dx$. Each of the second pair of integrals is covered by the examples in Question [1].

(b) Now you tackle $\int \frac{2x-3}{4x^2-20x+74}\,dx$. To get you going you will need to split the numerator as $2x-3 = (2x-5)+2$. AND you will have to complete the square to do the second integral!!

[4](2 marks) Another multistep integral: $\int \frac{e^x}{e^x+e^{-x}}\,dx$

Here is one approaches: substitute $u = e^x$ then do some algebra to get an integral of the last type in Question [1].

[5](0 marks) Let's finish off with four integrals from the December 1994 examination. You should now have all the tools for these examples.

(a) $\int_4^9 \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$ (b) $\int_0^{\ln 2} xe^{2x}\,dx$ (c) $\int x\sec^2 x\,dx$ (d) $\int \frac{t+1}{\sqrt{1-9t^2}}\,dt$

Solutions for Tutorial # 7

[1] Differentiate the right hand sides to get the integrands. I hope they all checked out.

[2] (a) For $\int \tan^{-1} x\,dx$, set $u = \tan^{-1}\ x$ and $dv = dx$. Then $du = \dfrac{1}{1+x^2}dx$ and $v = x$. Therefore

$$\int \tan^{-1} x\,dx = x\ \tan^{-1}\ x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ \ln(1+x^2) + C.$$

(b) Since we are going to use parts it is much simpler to do the indefinite integral first then go back to the definite integral. For $\int \sqrt{x}\ \ln\ x\,dx$ set $u = \ln\ x$ and $dv = \sqrt{x}\,dx$. Hence $du = \dfrac{1}{x}dx$ and $v = \dfrac{2}{3}x^{3/2}$. So

$$\begin{aligned}\int \sqrt{x}\ \ln\ x\,dx &= \frac{2}{3}x^{3/2}\ \ln\ x - \frac{2}{3}\int x^{3/2}\frac{1}{x}\,dx = \frac{2}{3}x^{3/2}\ \ln\ x - \frac{2}{3}\int x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2}\ \ln\ x - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right) + C = \frac{2}{3}x^{3/2}\ \ln\ x - \frac{4}{9}x^{3/2} + C.\end{aligned}$$

Hence $\displaystyle\int_4^{25} \sqrt{x}\ \ln\ x\,dx = \left[\frac{2}{3}x^{3/2}\ \ln\ x - \frac{4}{9}x^{3/2}\right]_4^{25} = \frac{2}{3}(125)\ln(25) - \frac{4}{9}(125) - \frac{2}{3}(8)\ln(4) + \frac{4}{9}8.$

[3] (a) Using the formulas in [1] we get

$$\int \frac{4x+5}{x^2+9}\,dx = 2\int \frac{2x}{x^2+9}\,dx + 5\int \frac{1}{x^2+9}\,dx = 2\ \ln(x^2+9) + 5\frac{1}{3}\ \tan^{-1}\left(\frac{x}{3}\right) + C.$$

(b) In the same spirit as the last integral but with more details to work out, we have

$$\begin{aligned}\int \frac{2x-3}{4x^2-20x+74}dx &= \int \frac{2x-5}{4x^2-20x+74}dx + \int \frac{2}{4x^2-20x+74}dx \\ &= \frac{1}{4}\ln(4x^2-20x+74) + \int \frac{2}{7^2+(2x-5)^2}\,dx\end{aligned}$$

In that last step we used $\int \dfrac{f'(x)}{f(x)}\,dx = \ln(f(x))$ in the first integral and in the second completed the square. In this second integral we can set $u = 2x-5$ so $du = 2dx$. It then becomes

$$\int \frac{2}{7^2+(2x-5)^2}\,dx = \int \frac{1}{7^2+u^2}\,du = \frac{1}{7}\tan^{-1}\left(\frac{u}{7}\right) = \frac{1}{7}\tan^{-1}\left(\frac{2x-5}{7}\right).$$

So finally, $\displaystyle\int \frac{2x-3}{4x^2-20x+74}dx = \frac{1}{4}\ln(4x^2-20x+74) + \frac{1}{7}\tan^{-1}\left(\frac{2x-5}{7}\right).$

[4] Let $u = e^x$ so $du = e^x\,dx$. Then

$$\int \frac{e^x}{e^x+e^{-x}}\,dx = \int \frac{1}{u+\frac{1}{u}}\,du = \int \frac{u}{u^2+1}du = \frac{1}{2}\ln(1+u^2) + C = \frac{1}{2}\ln(1+e^{2x}) + C.$$

[5] (a) We take $u = \sqrt{x}$ so $du = \dfrac{1}{2\sqrt{x}}dx$. Then

$$\int_{x=4}^{x=9} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx = \int_{u=2}^{u=3} \sin(u)\, 2\, du = -2\ \cos\ u\Big|_{u=2}^{u=3} = -2\cos\ 3 + 2\ \cos\ 2.$$

(b) $\displaystyle\int_0^{\ln\ 2} xe^{2x}\, dx$. This is by parts with $u = x, dv = e^{2x}\, dx$ and so $du = dx$ and $v = \dfrac{1}{2}e^{2x}\, dx$.

$$\int_0^{\ln 2} x\, e^{2x}\, dx = \frac{1}{2}x\, e^{2x}\Big|_0^{\ln 2} - \int_0^{\ln 2} \frac{1}{2}e^{2x}\, dx = \frac{1}{2}x\, e^{2x}\Big|_0^{\ln 2} - \frac{1}{4}e^{2x}\Big|_0^{\ln 2} = 2\ \ln\ 2 - \frac{3}{4}.$$

(c) PARTS WRITTEN IN A COMPACT WAY We are (silently) using $u = x$, $dv = \sec^2 x\, dx$.

$$\int x\ \sec^2\ x\, dx = \int x\ d(\tan x) = x\ \tan\ x - \int \tan\ x\, dx = x\ \tan\ x - \ln(\sec\ x) + C.$$

(d) Split it into two integrals.

$$\int \frac{t+1}{\sqrt{1-9t^2}}\, dt = \int \frac{t}{\sqrt{1-9t^2}}\, dt + \int \frac{1}{\sqrt{1-9t^2}}\, dt = \frac{-1}{9}\sqrt{1-9t^2} + \frac{1}{3}\sin^{-1}(3t) + C.$$

Elementary Calculus

Tutorial 8

Today: partial fractions, simple DE's and a volume.

[1](3 marks) For each of the following quadratics factor it into linear factors if possible. If it is not possible then complete the square. (The key : $ax^2 + bx + c$ factors exactly when $b^2 - 4ac \geq 0$.)

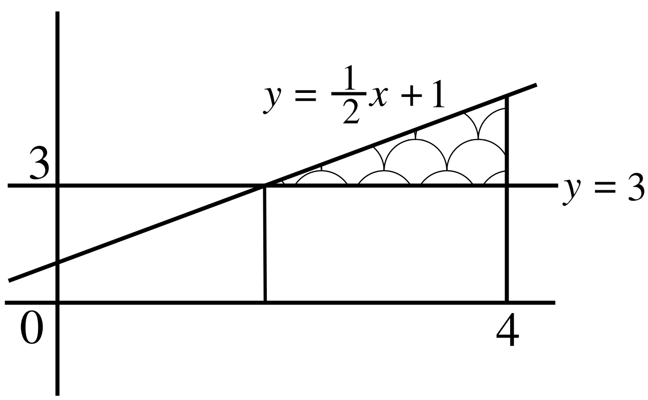
(a) $x^2 - 2x - 3$ (b) $x^2 - 2x + 5$ (c) $2x^2 + 21x - 11$

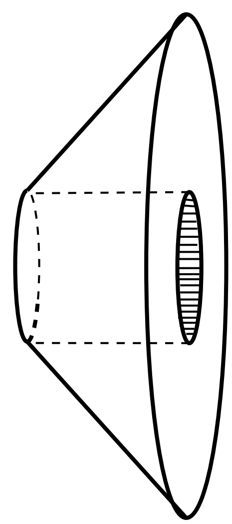
[2](3 marks)

(a) $\displaystyle\int \frac{x+2}{x^2 - 2x - 3}\,dx$ (b) $\displaystyle\int \frac{x+2}{x^2 - 2x + 5}\,dx$ (c) $\displaystyle\int \frac{x^3 - 2x^2 - 2x + 2}{x^2 - 2x - 3}\,dx$

[3](2 marks) Find a function y that satisfies $x^2 \dfrac{dy}{dx} = y$, with $y = e^2$ when $x = 3$.

[4](2 marks) The region bounded by the lines $y = \dfrac{1}{2}x + 1$, $y = 2$ and $x = 4$ is sketched below. Also I have tried to sketch the solid you would get by revolving this region about the x-axis (its rather like a tap washer).





(a) Calculate the x-coordinate of the point of intersection.

(b) Calculate the volume of the solid of revolution.

Stretchers

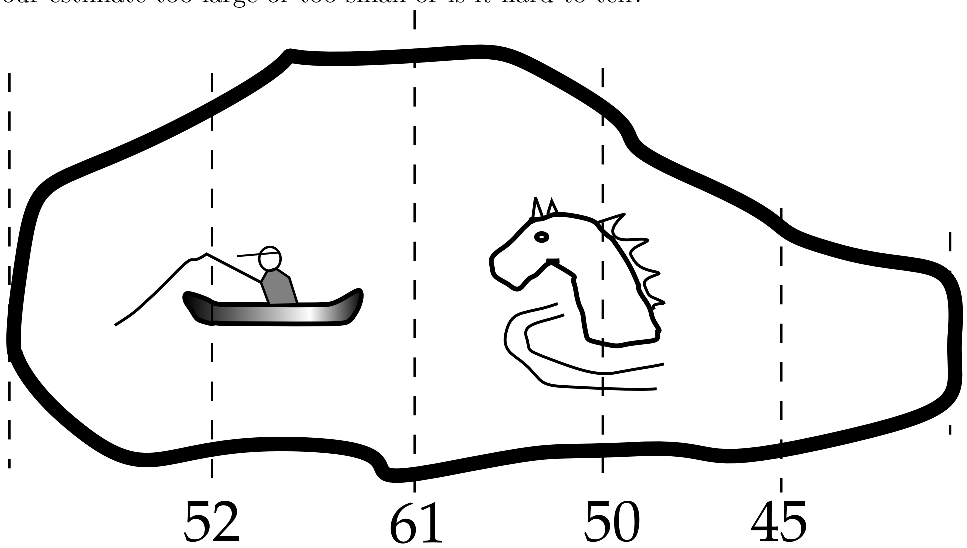
[5](0 marks) Tutorial # 3 began: Toxic mercury is pouring into a lake ... The concentration $C(t)$ at time t satisfies a differential equation:

$$\frac{dC(t)}{dt} = a + bC(t)$$

where a and b are constants. This is a separable DE. Now you are in a position to solve. So—solve it! (It may help to write $C(t)$ as y.)

[6](0 marks) $\displaystyle\int \frac{2x^4 + 21x^3 - 9x^2 + 22x - 9}{2x^2 + 21x - 11}\,dx$

[7](0 marks) This is a numerical integration question. I've stuck it in because I had fun with the picture! We want to estimate the area of the flat bottomed lake below. The surveyor measured the width at 30 m. intervals; the results are marked on the picture. Conveniently the lake is 150m long. Estimate the area. Is your estimate too large or too small or is it hard to tell?



Solutions for Tutorial #8

[1]

(a) $x^2 - 2x - 3 = (x-3)(x+1)$ $[b^2 - 4ac = 16]$

(b) $x^2 - 2x + 5 = (x-1)^2 + 2^2$ $[b^2 - 4ac = -16]$

(c) $2x^2 + 21x - 11 = (2x-1)(x+11)$ $[b^2 - 4ac = 23^2]$.

[2] (a) Set $\dfrac{x+2}{x^2-2x-3} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$. Take a common denominator on the right. The numerators of right and left sides must now be equal. $x+2 = (x+1)A + (x-3)B$. Substitute $x = 3$ to discover that $A = \dfrac{5}{4}$. Subsitute $x = -1$ to get $B = 1\dfrac{1}{4}$.

$$\int \frac{x+2}{x^2-2x-3}\,dx = \int \frac{5}{4}\frac{1}{x-3}\,dx + \int \left(-\frac{1}{4}\right)\frac{1}{x+1}\,dx = \frac{5}{4}\ln(x-3) - \frac{1}{4}\ln(x+1) + C$$

(b) Write $\dfrac{x+2}{x^2-2x+5} = \dfrac{x-1+3}{x^2-2x+5} = \dfrac{1}{2}\dfrac{2x-2}{x^2-2x+5} + 3\dfrac{1}{(x-1)^2+2^2}$. What we have done is produce one term with a constant numerator (the second) and in the other the numerator is the *derivative* of the denominator.

$$\int \frac{x+2}{x^2-2x+5}\,dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}\,dx + 3\int \frac{1}{(x-1)^2+2^2}\,dx = \frac{1}{2}\ln(x^2-2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

(c) Divide the numerator by the denominator to get $\dfrac{x^3-2x^2-2x+2}{x^2-2x-3} = x + \dfrac{x+2}{x^2-2x-3}$. Rejoice that you have already done the rest of the work.

$$\int \frac{x^3-2x^2-2x+2}{x^2-2x-3} = \int x\,dx + \int \frac{x+2}{x^2-2x-3}\,dx = \frac{1}{2}x^2 + \frac{5}{4}\ln(x-3) - \frac{1}{4}\ln(x+1) + C.$$

[3] The equation $x^2\dfrac{dy}{dx} = y$ can be rearranged to put a the x stuff on one side and all the y stuff on the other: $\dfrac{1}{y}\,dy = \dfrac{1}{x^2}\,dx$. Then integrate both sides to get $\ln y = -\dfrac{1}{x} + C$. We are told that $(x,y) = (3, e^2)$ satisfies the equation so we have $\ln e^2 = -\dfrac{1}{3} + C$. Hence $C = \dfrac{7}{3}$. The final solution is

$$y = e^{-\frac{1}{x}+\frac{7}{3}}.$$

[4] (a) The intersection point is at $x = 2$. Just set $\dfrac{1}{2}x + 1 = 2$.

(b) The volume is

$$\begin{aligned} VOL &= \pi\int_2^4 \left(\frac{1}{2}x+1\right)^2 - 2^2\,dx \\ &= \pi\left[\frac{2}{3}(\frac{1}{2}x+1)^3 - 4x\right]_2^4 \\ &= \frac{14\pi}{3}. \end{aligned}$$

Elementary Calculus

Tutorial 9

> The last tutorial: these are typical examination problems that we have not had time to review this week. We won't be marking this one but there are solutions available.

[1] First an integral to get you started. Evaluate $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$. [*Hint:* Set $u = \sqrt{1-x^2}$ so $u^2 = 1 - x^2$ and $x^2 = 1 - u^2$ and $du = ?$.]

[2] (i) Approximate $\int_0^8 \frac{x^2}{x+1}\,dx$ using the trapezoidal (we called it the average) method of numerical integration. Use four equal subintervals.

(ii) Evaluate the same integral with an antiderivative.

[3] Let $f(x) = \frac{(x-9)x}{6(x+1)}$. The derivative is $f' = \frac{x^2+2x-9}{6\,(x+1)^2}$.

(i) Find the vertical asymptote(s).

(ii) Determine the domain of $f(x)$ and the intervals on which $f(x)$ is positive or is negative.

(iii) Calculate the limits as $x \to \pm\infty$. (You may want to use L'Hopital's Rule.)

(iv) Determine the critical points of f, the intervals of increase and decrease. Identify the local maxima and minima. Justify your answers. [For this function the answers to this part are not rational numbers. You should run into $-1 \pm \sqrt{10}$ along the way.]

(v) Sketch the graph.

Stretchers

[4] (i) Calculate the derivative of $y = \ln(\sec x + \tan x)$. (ii) Evaluate $\int \sec x\,dx$.[The two parts are not unrelated!!]

[5] Here is a bit of classical math. As you know $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$.

(i) Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$.

(ii) On the other hand, we can use an infinite geometric series to write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$$

and hope we can get away with

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^1 1\,dx - \int_0^1 x^2\,dx + \int_0^1 x^4\,dx - \int_0^1 x^6\,dx + \int_0^1 x^8\,dx \cdots$$

Hence deduce the classical formula $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$ due to Jakob Bernoulli in 1696.

Solutions for Tutorial #12

[1] As proposed, set $u = \sqrt{1-x^2}$ so $du = \dfrac{-x}{\sqrt{1-x^2}}\,dx$ and also $x^2 = 1 - u^2$. Then

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int (-x^2)\ \frac{-x}{\sqrt{1-x^2}}\,dx = \int u^2 - 1\,du = \left[\frac{1}{3}u^3 - u\right] + C = \frac{1}{3}(1-x^2)^{3/2} - (1-x^2)^{1/2} + C.$$

[2] (i) The ends of the intervals are 0, 2, 4, 6, 8. The values of the function $f(x) = \dfrac{x^2}{x+1}$ at these points are $f(0) = 0$, $f(2) = 1.333$, $f(4) = 3.2$, $f(6) = 5.1429$, $f(8) = 7.111$. The averages on the intervals are 0.6667, 2.2667, 4.1715 and 6.1270. So

$$\int_0^8 f(x)\,dx \approx 0.6667 \times 2 + 2.2667 \times 2 + 4.1715 \times 2 + 6.1270 \times 2 = 26.46.$$

The 2's in the last formula are the lengths of the intervals.

(ii) Using an antiderivative we get

$$\int_0^8 \frac{x^2}{x+1}\,dx = \int_0^8 \frac{x^2-1+1}{x+1}\,dx = \int_0^8 x - 1 + \frac{1}{x+1}\,dx = \left[\frac{1}{2}x^2 - x + \ln(x+1)\right]_0^8 = 26.197.$$

[3] (i) The vertical asymptotes occur at the places where the denominator is zero. Hence at $x = -1$.

(ii) The function $f(x)$ is a defined for att $x \neq -1$. The function changes sign at $x = 9, 0$ and -1. So $f > 0$ for $-1 < x < 0$ and $9 < x$. And similarly $f < 0$ for $x < -1$ and $0 < x < 9$.

(iii) $\lim_{x\to\infty} f(x) = \infty$ and also $\lim_{x\to-\infty} f(x) = -\infty$.

(iv) The roots of $f' = 0$ are the roots of the numerator $x^2 + 2x - 9$. Thus the critical points are at $-1 \pm \sqrt{10} = 2.16$ and -4.16. Since $(x+1)^2 \geq 0$, the derivative changes sign only at these roots and behaves signwise like the parabola in the numerator. Thus f' changes from positive to negative as x moves up through -4.16 and changs from negative to positive at 2.16. Therefore there is a local maximum at -4.16 and a local minimum at 2.16.

(v) The graph looks like this: