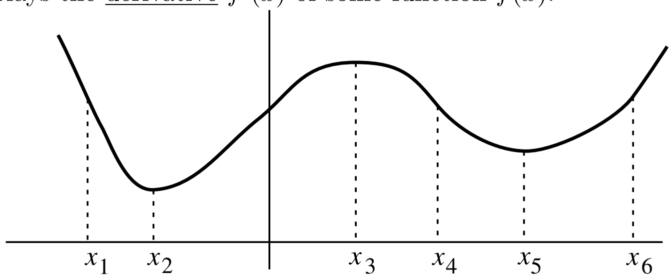
Elementary Calculus

Assignment #1

[1] Calculate the derivative $\frac{dy}{dx}$ given that:

(a) $y = \tan x \sqrt{\sin x}$ (b) $\tan(xy) = x^2 + y$

[2] The graph below displays the derivative $f'(x)$ of some function $f(x)$.



At which of the marked values of x is:

(a) $f(x)$ largest, (b) $f(x)$ smallest, (c) $f'(x)$ largest,

(d) $f'(x)$ smallest, (e) $f''(x)$ largest, (f) $f''(x)$ smallest?

Justify your answer in sentences.

[3] (Final Examination, Dec. 1994) Consider the function $y = f(x) = \dfrac{x-4}{x+1}e^{x+1}$.

(a) Find the vertical asymptotes of the function.

(b) For what values of x is the function positive and for what values is it negative?

(c) Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(d) Show that $\frac{dy}{dx} = f'(x) = \frac{x^2-3x+1}{(x+1)^2}e^{x+1}$

(e) Find all critical points of the function.

(f) Determine the intervals on which the function is increasing, decreasing.

(Hint: make a table of signs for the derivative.)

(g) Investigate all local maximum and minimum values. Use an appropriate test.

(h) Sketch the graph $y = f(x)$.

[4] (a) Sketch the graphs of $y = e^{-x}$ and $y = \cos x$ on the same axes.

(b) Use Newton's Method to approximate (to three decimal places) the smallest value of $x > 0$ where these curves intersect.

Elementary Calculus

Assignment #2

[1] Consider the following model for the growth of a city. The shape of the city always remains roughly circular so that the maximum travel time between two locations in the city is proportional to the diameter of the city. The population of the city is proportional to the area of the city. The rate of increase of the city's population is inversely proportional to the maximum travel time.

(i) This model predicts that the population of the city $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = \frac{K}{\sqrt{P(t)}}$$

where K is constant. Explain why this DE expresses the information in the paragraph.

(ii) Verify (by differentiating) that $P(t) = (\frac{3}{2}Kt + C)^{2/3}$ is a solution of the diff. equation in part (i).

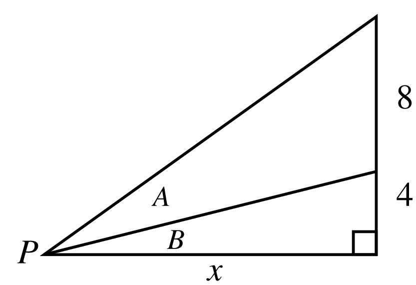
(iii) If the population of the city was 5000 in 1900 and 20,000 in 1959, what is the predicted population of the city in the year 2000?

[2] For each of the following functions $f(x)$, determine its <u>domain</u> and calculate its <u>derivative</u>.

(a) $f(x) = \sin^{-1}(\ln x)$ (b) $f(x) = \tan^{-1}(\frac{1}{\sqrt{x}})$

[3] Using Inverse Trig. Functions.

(a) In the diagram, as the length x increases the point P moves to the left and angle A changes. Determine a formula $f(x)$ such that $A = f(X)$. [Hint: use $\tan^{-1}$ to get formulas for angles B and $A + B$ then subtract.]



(b) Determine the maximum value of $f(x)$ for $x \geq 0$.

[4] Consider the family of functions $y = a + b\,e^{x^2}$ where a and b are parameters. Determine the values of a and b such that y is a solution of the differential equation

$$\frac{dy}{dx} = x + 2xy.$$

[5] Determine an antiderivative of $\dfrac{x}{1+x^2}$ and hence evaluate $\displaystyle\int_0^{16} \frac{x}{1+x^2}\,dx$.

[1] (a) For simplicity let P be the population, M be the maximum travel time, A the area of the city and d be its diameter. Then P is proportinal to A and A is proportional to d^2 so P is proportional to d^2. Then M is proportional to d and P is proportional to d^2 so M is proportional to $\sqrt{P}$. (You could also say all this using constants to express the proportions.) But P' is proportional to $\frac{1}{M}$ so finally P' is proportional to $\frac{1}{\sqrt{P}}$. We can therefore write

$$\frac{d\,P(t)}{d\,t} = \frac{K}{\sqrt{P(t)}}$$

(b) With $P(t) = (\frac{3}{2}Kt + C)^{2/3}$ we have

$$P'(t) = \frac{2}{3}\left(\frac{3}{2}Kt + C\right)^{-\frac{1}{3}}\left(\frac{3}{2}K\right) = \frac{K}{(\frac{3}{2}Kt + C)^{\frac{1}{3}}} = \frac{K}{\sqrt{(\frac{3}{2}Kt + C)^{\frac{2}{3}}}} = \frac{K}{\sqrt{P(t)}}.$$

(c) Measure the population in kilopeople and time starting with $t = 0$ in 1900. Then we have $P(0) = 5 = C^{2/3}$ so $C = \sqrt{125}$. Next $P(59) = 20 = (\frac{3}{2}K59 + \sqrt{125})^{2/3}$ so $K = 0.88$. Finally we want $P(100) = (\frac{3}{2}(0.88)100 + \sqrt{125})^{2/3} = 27.4$ kilopeople. If you used different units, your values for K and C will differ but the final answer should be the same.

[2] (a) $f(x) = \sin^{-1}(\ln x)$ is defined when $-1 \le \ln x \le +1$ hence for $e^{-1} \le x \le e$. The derivative is

$$f'(x) = \frac{1}{\sqrt{1 - (\ln x)^2}}\frac{1}{x}$$

(b) The function $\tan^{-1} u$ can be calculated for any number u. Thus the domain of $\tan^{-1}(\frac{1}{\sqrt{x}})$ is the set of all x where we can calculate the square root. So $x > 0$ is the domain.

$$f'(x) = \frac{1}{1 + (\frac{1}{\sqrt{x}})^2} \times (-\frac{1}{2}) \times \frac{1}{x^{3/2}} = \frac{-1}{2(x+1)x^{1/2}}.$$

[3] (a) The full right hand side is 12 units high so $\tan(A + B) = 12/x$. We can write this as $A + B = \tan^{-1}(12/x)$. Similarly $B = \tan^{-1}(4/x)$. Finally we get

$$A = f(x) = \tan^{-1}(12/x) - \tan^{-1}(4/x).$$

(b) The derivative of $f(x)$ is

$$f'(x) = -8\frac{x^2 - 48}{(x^2 + 144)(x^2 + 16)}.$$

The derivative is > 0 for $x < \sqrt{48}$, is $= 0$ for $x = \sqrt{48}$ and is < 0 for $x > \sqrt{48}$. By the first derivative test there is a local maximum at $x = \sqrt{48}$. So the maximum value of the angle is $f(\sqrt{48}) = \tan^{-1}(12/\sqrt{48}) - \tan^{-1}(4/\sqrt{48}) = 0.5235\text{rad} = 30°$.

[4] We want to know which of the functions $y = a + b\,e^{x^2}$ satisfy $\frac{d\,y}{d\,x} = x - 2xy$. Differentiating the formula $y = a + b\,e^{x^2}$ we get $y' = 2bxe^{x^2}$. We want to determine a and b so that this is $x - 2xy = x - 2x(a + b\,e^{x^2}) = (1 - 2a)x - 2bxe^{x^2}$. The only way to have $2bxe^{x^2} = (1 - 2a)x - 2bxe^{x^2}$ for every x is to have $a = \frac{1}{2}$ and $b = 0$.

I *meant* to use the differential equation $\frac{d\,y}{d\,x} = x + 2xy$.. In this case we solve the problem as follows: We want to know which of the functions $y = a + b\,e^{x^2}$ satisfy $\frac{d\,y}{d\,x} = x + 2xy$. Differentiating the formula $y = a + b\,e^{x^2}$ we get $y' = 2bxe^{x^2}$. We want to determine a and b so that this is $x + 2xy = x + 2x(a + b\,e^{x^2}) = (1 + 2a)x + 2bxe^{x^2}$. To have $2bxe^{x^2} = (1 + 2a)x + 2bxe^{x^2}$ for every x we must have $a = -\frac{1}{2}$ but b can have any value; there are infinitely many solutions.

[5] One antiderivative of $\dfrac{x}{1 + x^2}$ is $\frac{1}{2}\,\ln(1 + x^2)$. So the exact value of the definite integral is

$$\int_0^{16} \frac{x}{1 + x^2}\,dx = \frac{1}{2}\,\ln(1 + (16)^2) - \frac{1}{2}\,\ln(1 + (0)^2) = \frac{1}{2}\,\ln 257.$$

Elementary Calculus

Assignment #3

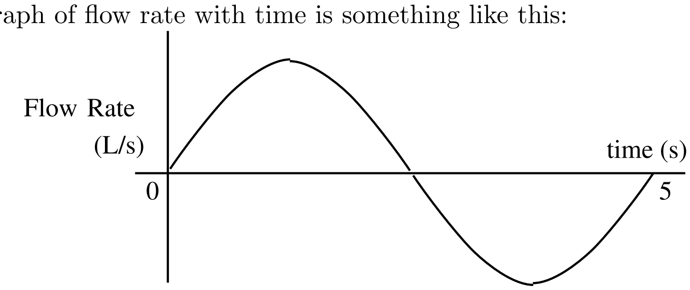
Please remember, wherever it is sensible, to write your answers in *sentences* that help the reader (i.e. marker) follow your answers.

[1] Make a sketch of the graphs of $y = x^3 - x$ and $y = 3x$ on the same axes. Calculate the area between the curves.

[2] Calculate the following definite and indefinite integrals.

(a) $\displaystyle\int_4^9 \frac{1}{\sqrt{x}(\sqrt{x}-1)^2}\,dx$ (b) $\displaystyle\int_0^\pi x\sin 5x\,dx$ (c) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ (d) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$

[3] Normally air moves smoothly in and out of a persons lungs. Observations show that a complete cycle from the start of one in-breath to the start of the next is 5 sec. Also the maximum flow rate of air moving into the lungs is about 0.05 L/sec. The flow rate is positive as air moves in and is negative as air is exhaled. So the graph of flow rate with time is something like this:



(a) Since this graph looks like a sine function it is reasonable to model the flow rate with a function of the form $f(t) = a\sin\left(\frac{2\pi}{b}t\right)$. Determine the values of the constants a and b that will fit the function $f(t)$ to the data.

(b) Use the model calculated in (a) to estimate the total volume of air moving into the lungs in one breath. (This will involve a definite integral.)

[4] Use the Fundamental Theorem of Calculus to calculate the derivative of the function

$$F(x) = \int_{x^2}^{0} \sqrt{\sin t}\,dt.$$

[*Hint*: $F(x) = -G(x^2)$ where $G(u) = \displaystyle\int_0^u \sqrt{\sin t}\,dt$.] (Final Exam, April '91)

Elementary Calculus

Assignment #4

[1] (a) Sketch the region bounded by $y = x^2 + 3$ and $y = x^{\frac{1}{3}}$ between $x = 1$ and $x = 2$.

(b) Calculate the volume of the solid obtained by revolving this region about the x-axis.

[2] Calculate $\int \frac{2x^3-7x^2+39x+26}{x^2-4x+21}\,dx$.

[3] Calculate the solution of the differential equation $\frac{dy}{dx} = \frac{x+e^x}{3y^2}$ that passes through the point $(x, y) = (0, 3)$.