Interlude: Linear Program Formulations for Some Combinatorial Optimization Problems

Linear Programming is a commonly taught and used form of optimization. In this area, one tries to maximize or minimize a linear objective function subject to linear inequality contraints. In fact some Combinatorial Optimization problems can be recast as LP problems which can then be solved by standard LP algorithms. In some cases this is a fruitful approach, in others the number of constraints required is so large that solving the problem as an LP is impractical. In this brief interlude we will discuss how two combinatorial optimization problems can be solved through an associated linear program.

Let $G = (V, E)$ be a digraph where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices, and $E = \{e_1, e_2, \ldots, e_m\}$ is the set of arcs. The *incidence matrix* M of G is defined as follows. The rows of M are indexed by the vertices, and the columns of M are indexed by the arcs. The entries of M are all ± 1 or 0. If $e_k = (ij)$ joins v_i to v_j, then $M_{ik} = -1$ and $M_{jk} = 1$. In the following, each arc (ij) is assigned a weight w_{ij}.

Potentials and the single source Shortest Path problem

Consider the problem of finding a minimum weight dipath from a source vertex s to other vertices of G.

A vector $\vec{g} = [g_1, g_2, \ldots, g_n]$ is called a *potential* with respect to the weight $\vec{w}$ if

$$g_j - g_i \leq w_{ij} \tag{1}$$

Note that the above system of inequalities (1) can be rewritten using the incidence matrix M as

$$\vec{g}M \leq \vec{w}$$

For each $v \in V$, let y_v be the weight of a minimum dipath from the source node s to a vertex v. Suppose $P\colon s = v_0, v_1, \ldots, v_k = v$ is a dipath with weight y_v. Then we have

$$y_v = \sum_{j=0}^{k-1} w_{v_j v_{j+1}} \geq \sum_{j=0}^{k-1} \left(g_{v_{j+1}} - g_{v_j} \right) = g_v - g_s$$

On the other hand, $\vec{y}$ itself is a potential (Exercise) and trivially $y_s = 0$, so

$$y_v = y_v - y_s \geq g_v - g_s$$

for any potential $\vec{g}$. Hence the value of y_v is given by the maximum potential.

Define a vector $\vec{c}$ indexed by vertices with $c_v = 1$, $c_s = -1$, and $c_u = 0$ for all $u \in V - \{v, s\}$. So $y_v - y_s = \vec{y}\vec{c}$ and $g_v - g_s = \vec{g}\vec{c}$. Then the following LP:

$$\text{Maximize } \vec{g}\vec{c} \text{ subject to } \vec{g}M \leq \vec{w} \tag{2}$$

has as its value the length y_v of a minimal dipath from s to v. In the LP there is one constraint for each arc. If the graph is reasonably sparse then this approach may well lead to an efficient solution to the minimal dipath problem.

The dual of the above LP is, with variable vector $\vec{h}$ indexed by arcs:

$$\text{Minimize } \vec{w}\vec{h} \text{ subject to } M\vec{h} = \vec{c}, \quad \vec{h} \geq 0. \tag{3}$$

Each dipath P from s to v gives a feasible solution $\vec{h}_P$ to this dual LP, which is called the *characteristic vector* of P. It is defined by $h_e = 1$ for each arc $e \in P$, and $= 0$ otherwise. Then $\vec{w}\vec{h}_P$ is the weight of the dipath P. What LP Duality Theory tells us is that the minimum value of the dual LP is equal to the maximal value of the primal LP and this value is the weight of a minimal dipath from s to v. The minimal value of the dual LP is achieved by the characteristic vector of a minimal dipath. There may be other optimal solutions of the dual that are not characteristic vectors of dipaths.

We note though that an arbitrary vector $\vec{h}$ with components 0 or 1 which is dual feasible is a characteristic vector of an $s - v$ dipath. Indeed the matrix product $M\vec{h} = \vec{c}$ can be considered a set of equations – one for each vertex u. The equation for this vertex (using the definition of M) adds the components of $\vec{h}$ on arcs with u as head and subtracts the components with u as tail. The definition of the vector $\vec{c}$ (value $+1$ at v , -1 at s and 0 elsewhere) ensures that if $\vec{h}$ is a $0-1$ vector feasible for the dual LP, then $\vec{h}$ is a characteristic vector of a dipath from s to v.

This discussion shows how the minimum path problem and the maximum potential problem can be formulated as linear programs that are dual to each other.

Maxflow and Mincut problems

A network consists of a digraph G, a source node s and a sink node t in G, and a capacity function $c = \{c_e : e \in E\}$. A vector $f = \{f_e : e \in E\}$ is called a flow in G if

$$\sum_{h(e)=v} f_e - \sum_{t(e)=v} f_e = 0, \quad v \in V - \{s, t\} \tag{4}$$

where $t(e)$ and $h(e)$ denote the tail and head of the arc e, respectively. Condition (4) is called the *flow conservation law*. Note that the flow conservation law can be expressed as

$$\hat{M}f = 0$$

where $\hat{M}$ is the matrix obtained from the incidence matrix M by deleting rows corresponding to s and t. The maximum flow problem can be formulated as the following linear program.

$$\text{Maximize } \sum_{h(e)=t} f_e - \sum_{t(e)=t} f_e \text{ subject to } \hat{M}f = 0, \quad 0 \leq f \leq c$$

The augmenting path algorithm is a primal algorithm, since at each step it maintains a feasible flow and improves towards getting a dual feasible solution (a saturated cut).

The preflow-push algorithm can be viewed as a dual algorithm, since at each step it mains a dual feasible solution (saturated cut), and improves towards getting a primal feasible solution (feasible flow).

The LP formulation of transportation and assignment problems is straightforward. One usually uses a primal (simplex) algorithm to solve the transportation problem, and uses a dual algorithm to solve the assignment problem (such as Kuhn's Hungarian method).