Carleton UNIVERSITY
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Mathematics Tutorial Series

L'Hospital's Rule

L'Hospital (or L'Hôpital) included this rule in the first text on Differential Calculus in 1696.

It is about limits called "indeterminate forms".

Essentially these are built from elements that pull irresistibly in different directions.

Suppose we want to evaluate a limit that looks like $\frac{0}{0}$

For example

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-1}$$

$$\lim_{x\to \pi/4} \frac{\sin x - \cos x}{x-\frac{\pi}{4}}$$

L'Hospital's Rule for $\frac{0}{0}$

If $\lim_{x\to a} f(x) = \lim_{\mathrm{x}\to \mathrm{a}} g(x) = 0$ and

$\lim_{\mathrm{x}\to \mathrm{a}} \frac{f'(x)}{g'(x)}$ has a value

then $\lim_{\mathrm{x}\to \mathrm{a}} \frac{f(x)}{g(x)} = \lim_{\mathrm{x}\to \mathrm{a}} \frac{f'(x)}{g'(x)}$

Also works

- For $\frac{\infty}{\infty}$ forms like: $\lim_{x\to\infty} \frac{\log x}{e^x}$
- And if $\lim_{\mathrm{x}\to \mathrm{a}} \frac{f'(x)}{g'(x)} = \pm\infty$
- And if the limit is as $x \to \pm\infty$

Examples:

Example 1:

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2-1}=\lim_{x\to 1}\frac{(\sqrt{x}-1)'}{(x^2-1)'}=\lim_{x\to 1}\frac{\frac{1}{2\sqrt{x}}}{2x}=\frac{1}{4}$$

Example 2:

$$\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}=\lim_{x\to\frac{\pi}{4}}\frac{\cos x+\sin x}{1}$$

$$=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}=\sqrt{2}$$

Example 3:

$$\lim_{x\to\infty}\frac{\log x}{e^x}=\lim_{x\to\infty}\frac{(\log x)'}{(e^x)'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{e^x}=0$$

Example 4:

$$\lim_{x\to 1}\frac{\log x}{x-1}=\lim_{x\to 1}\frac{(\log x)'}{(x-1)'}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=1$$

ERROR – Non-examples:

What's wrong here?

$$\lim_{x\to 3}\frac{x-3}{x^2-3}=\lim_{x\to 3}\frac{1}{2x}=\frac{1}{6}$$

and here?

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Example 5:

$$\lim_{x\to 0}x\log x$$

This is indeterminate since the limit appears to be $0\times\infty$

Rewrite this as an $\frac{\infty}{\infty}$:

$$\lim_{x\to 0}\frac{\log x}{\frac{1}{x}}=\lim_{x\to 0}\frac{1/x}{-1/x^2}=\lim_{x\to 0}-x=0$$

Example 6:

$$\lim_{x \to 0} x^x$$

This is appears to have a limit like 0^0 which is indeterminate. Let $y = x^x$ and take $\lim_{x\to 0} \log y$.

$$\lim_{x \to 0} \log y = \lim_{x \to 0} x \log x = 0$$

But $\log y = 0$ if and only of $y = 1$ so

$$\lim_{x \to 0} x^x = 1$$

Caution: You can't use L'Hospital's Rule on the limits used in the calculation of derivatives you would then use to apply L'Hospital's Rule. So to avoid circular reasoning, we can't use the L'Hospital's Rule on these ones.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{h \to 0} \frac{r^h - 1}{h} = \log r$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Summary

1. L'Hospital's Rule is a simple way to calculate limits when it applies.

2. When it applies:

$$\lim_{\mathrm{x}\to\mathrm{a}} \frac{f(x)}{g(x)} = \lim_{\mathrm{x}\to\mathrm{a}} \frac{f'(x)}{g'(x)}$$

3. The limit has to be of the right type and $\lim_{\mathrm{x}\to\mathrm{a}} \frac{f'(x)}{g'(x)}$ must exist

4. You can use L'Hospital's Rule

- For $\frac{0}{0}$ forms like: $\lim_{x\to 0} \frac{\cos x - 1}{x}$
- For $\frac{\infty}{\infty}$ forms like: $\lim_{x\to\infty} \frac{\log x}{e^x}$
- And if $\lim_{\mathrm{x}\to\mathrm{a}} \frac{f'(x)}{g'(x)} = \pm\infty$
- And if the limit is as $x \to \pm\infty$