Carleton UNIVERSITY
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Mathematics Tutorial Series

Exponential Models III

Newton's Law of Cooling and financial models provide two more examples of exponential models.

Financial Models

If you invest $A(0)$ dollars at 3% per year then, each year, you increase your total holding by multiplying by $1+.03$. This gives you what you had plus the interest.

After t years you will have:

$$A(t) = A(0)(1.03)^t$$

This is an exponential model of investment growth.

It is a "model" because it ignores the compounding rule used by the investment holder.

Perhaps you are paid 1/12 of the interest rate each month or 1.5% every 6 months, for example.

We can ask all the same questions about this model:

- How much was there at $t = 0$?
- When will the amount be x?
- When was it y?
- How much will it be in z years?

Example 1:

For example we can ask, how long it will take to double our money.

$$\frac{A(t)}{A(0)} = (1.03)^t = 2$$

So the answer is $t = \frac{\log 2}{\log 1.03} = 23.45$ years.

Example 2:

We can sharpen the model:

Suppose interest is paid n times during the year with annual rate r. Then a model would be:

$$\mathrm{B(t)} = \mathrm{B(0)}\left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{nt}}$$

This is also an exponential model.

With r = 3%, n = 12 and t = 3 we get

B(3)=B(0)(1.09405)

Compare: A(3) = A(0)(1.09273)

Example 3:

And what is the present value of $50,000 invested for 3 years at 6%?

$$\$50{,}000 = A(0)(1.06)^3$$

So $A(0) = \frac{50000}{(1.06)^3} = \$41{,}981$

Example 4:

Suppose:

$$\$50{,}000 = \$42{,}000\ (1 + r)^3$$

We want to know the interest rate that will have $42,000 grow to $50,000 in three years. Solve for r.

$$r = \left(\frac{50000}{42000}\right)^{\frac{1}{3}} - 1 = 0.05984$$

Newton's Law of Cooling

Heat always flows from a hot place to a cold place. Transmission is by radiation, conduction or convection.

Newton's Law of Cooling says that the rate of cooling of an object is proportional to the difference between the temperature of the object and the surrounding environment. It is best to use the difference D between the temperatures as the variable since the external temperature is constant, the difference will change at the same rate as the temperature itself.

Newton's Law of Cooling says:

$$D' = kD$$

This differential equation has a solution we have been working with:

$$D(t) = D(0)e^{kt}$$

Since the difference will be decreasing the parameter $k < 0$.

Newton's Law of Cooling is valid for thermal transfer by convection. The air moving around a cup of coffee cools by convection. The other forms of transfer are conduction and radiation.

Typical example is a cup of coffee. Given enough time the coffee will cool to room temperature.

If we start with coffee at $63°\ C$ and the room temperature is $21°\ C$ then $D(0) = 42°$. If the difference drops to $30°$ in 5 minutes then the model is:

$$D(5) = 30 = 42e^{k5}$$

We measure time in minutes.

Take logs:

$$\log 30 = \log 42 + 5k$$

Then

$$k = \left(\frac{1}{5}\right)\log\frac{30}{42} = -0.0673$$

Then

$$D(t) = D(0)e^{-0.0673\,t}$$

Once we know this is an exponential decay situation and know that we should work with the temperature difference, the rest is like all the other exponential model problems.

Exercise: Convert the cooling model above to a half-life model and calculate that the half-life of the temperature difference is about 10 minutes.
(See Video App2b: Exponential Models II)

Summary

1. Exponential models are used in finance for example in calculating compound interest.

2. Newton's Law of Cooling works with the difference between the objects temperature and the surrounding temperature.

3. It is then an exponential decay model.