Carleton UNIVERSITY
Canada's Capital University

Mathematics Tutorial Series

Exponential Models II

Half-life and Doubling-time Models

The general exponential model has the formula:

$$A(t) = A(0)e^{kt}$$

When we use this for modeling it is advantageous to reformulate it to make the parameter k meaningful.

Half-life Models

"Half-life models" are a special formulation of exponential models.

For example, radioactive decay follows a half-life model. In these cases we specify a parameter λ in time units, say years. Then our model says that in λ years half of our sample will have experience radioactive decay, i.e. will be gone. In any given time t there are $\frac{t}{\lambda}$ half-life periods and, in each half-life period, half the amount disappears. Hence

$$A(t) = A(0)\left(\frac{1}{2}\right)^{\frac{t}{\lambda}}$$

Example 1.

Iron–55 (^{55}Fe) has a half-life of 2.7 years.
(a) How much of a 100 gm. sample will remain after 6 years?
(b) How long will it take for a 100 gm. sample of radioactive iron to decay to 20 gm.?

Solution

$$A(t) = A(0)\left(\frac{1}{2}\right)^{\frac{t}{\lambda}}$$

Measure time in years.

(a) From the formula: $A(6) = 100\left(\frac{1}{2}\right)^{\frac{6}{2.7}} = 21.43$ gm.

(b) Now t is unknown: $20 = 100\left(\frac{1}{2}\right)^{\frac{t}{2.7}}$.

So $0.2 = (0.5)^{\frac{t}{2.7}}$

Take logarithms: $\log 0.2 = \frac{t}{2.7}\log 0.5$

and so $t = 2.7\,\frac{\log 0.2}{\log 0.5} = 6.27$ years.

Carbon dating

2. A sample of wood is taken from the centrepost of an old building in Suffolk, England. It is estimated that this sample now has 89% of its original radioactive carbon-14 (^{14}C) content. If the half-life of carbon-14 is 5700 years, what is the expected age of the sample? (As a check on your calculations, you can use the knowledge that archeologists believe that the centrepost is at least 925 years old.)

Solution

Let $C(0)$ be the amount of carbon-14 at time $t = 0$, when the wood grew as a tree. Let $C(t)$ be the amount of carbon-14 at a later time t. The half-life formula is $C(t) = C(0)\left(\frac{1}{2}\right)^{\frac{t}{\lambda}}$ where λ is the half-life which is $\lambda = 5700$ years in our case. Our problem says that now $\frac{C(t)}{C(0)} = 0.89 = 89\%$. So we solve:

$$\frac{C(t)}{C(0)} = \left(\frac{1}{2}\right)^{\frac{t}{5700}} = .89$$

Take logs.

$$\frac{t}{5700}\log 0.5 = \log 0.89$$

and then

$$t = 5700\,\frac{\log 0.89}{\log 0.5}$$

$$t = 958.3$$

Thus the sample is about 958 years old.

3. In 1953, carbon-14 dating was used to calculate the age of a painting attributed to Vermeer who lived from 1632 to 1675. It was found that, through radioactive decay, only 99.5% of the original carbon-14 remained in the paint. Using the fact that the half-life of carbon-14 is 5700 years, calculate the approximate age of the painting and decide if there is evidence that the painting is a forgery. (Such a legal case did actually happen.)

4. A pesticide sprayed on tomatoes decomposes into a harmless substance at a rate proportional to the amount remaining. One week after the fields are sprayed, it is found that the initial amount of pesticide has decayed to one half its original amount. The Department of Agriculture considers that a safe level of the pesticide at harvest is 0.25 kilograms per hectare. The fields are sprayed 10 days before harvest time. Find the maximum amount of pesticide, which may be used on each hectare of land, if there is to be a safe level at harvest time.

Solution

In this problem we should work with time units set to days. It is a half-life problem with half-life $\lambda = 7$ days. With $P(0)$ as the amount of pesticide applied and $P(t)$ the amount at time t later, the formula is:

$$P(t) = P(0)\left(\frac{1}{2}\right)^{\frac{t}{7}}$$

In this problem we are given both $t = 10$ and $P(10) = 0.25$ kg/ha and are asked to find $P(0)$.

Then

$$\left(\frac{1}{2}\right)^{\frac{10}{7}} = 0.3715$$

and

$$P(0) = \frac{P(10)}{0.3715} = \frac{0.25}{0.3715} = 0.673$$

So the maximum amount that can be sprayed 10 days before harvest is 0.669 kg/ha.

Doubling Models

Some exponential models are described by giving the doubling time β. Thus we might say a population will double every $\beta = 11$ years or an investment may double every $\beta = 7$ years. Here we divide time into "doubling" units: $\frac{t}{\beta}$. The model is then:

$$A(t) = A(0)\,2^{\frac{t}{\beta}}$$

Relation to Exponential models

Every exponential model can be recast as a half-life model and also as a doubling-time model.

If $a(t) = a(0)e^{kt}$ then $a(t) = a(0)\, r^{\frac{k}{\log r}t}$
for any base $r > 0$.

So we can have: $a(t) = a(0)\left(\frac{1}{2}\right)^{\frac{kt}{\log 0.5}}$

Or we can have: $a(t) = a(0)2^{\frac{kt}{\log 2}}$

- Use half-life models for decay processes.
- Use doubling models for growth processes.

Summary

1. Half-life and doubling-time models are very common reformulations of general exponential models.

2. Half-life and doubling-time models allow us to specify the "k" in a way that makes sense in the world outside mathematics.

3. Every exponential model can be rewritten as a half-life or doubling-time model.