Carleton University
Canada's Capital University

Mathematics Tutorial Series

Exponential Models I
Exponential Models II – Half-life Models
Exponential Models III – Financial models and Newton Cooling

The general form of an exponential model is:

$$P(t) = P(0)e^{kt}$$

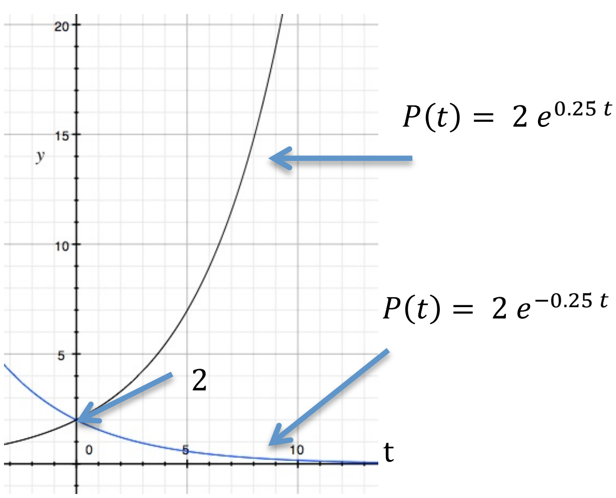
Since the derivative of $P(t)$ is $P(0)\,ke^{kt}$,
this function is a solution of the differential equation:

$$y' = ky.$$

"Exponential Model" refers to both the DE and its solution.

Some applications:

- Unconstrained growth
- Radioactive decay
- Thermal transfer by convection
- Pressure in the atmosphere
- Financial models



Before looking at some examples, lets look at the equation again.

$$P(t) = P(0)e^{kt}$$

A problem may require you to do any of the following:

i. Solve for t given k, $P(0)$ and $P(t)$
ii. Solve for t given k, and the ratio $P(t)/P(0)$
iii. Solve for $P(t)$ given k, t and $P(0)$
iv. Solve for $P(0)$ given k, t and $P(t)$
v. Solve for k given t, $P(0)$ and $P(t)$

If k and t are given then e^{kt} can be calculated. Then solving for one of $P(t)$ or $P(0)$ from the other is easy.

If k or t is what we are looking for we need a different technique.

Taking logs we get

$$\log P(t) = \log P(0) + \log e^{kt}$$

then

$$\log P(t) = \log P(0) + kt$$

This allows us to solve for one of k or t when the other quantities are known.

The original equation can be rewritten as

$$\frac{P(t)}{P(0)} = e^{kt}$$

This is how we work with the ratio.

Examples

1. Suppose $y(t)$ satisfies the differential equation

$$y' = 7y$$

and that $y(0) = 1000$. Find $y(8)$.

Solution

The function must have the form $y(t) = y(0)e^{kt}$ and we are given that $y(t)$ satisfies the equation

$$y' = 7y$$

So $k = 7$ and $y(0) = 1000$.

Hence the function is: $y(t) = 1000\, e^{7t}$.

We are asked to find:

$$y(8) = 1000\, e^{56} = 2.1 \times 10^{27}$$

2. When a population grows with no constraints, with no limits imposed by resources, the model used is exponential. Bacteria grow this way as long as there is an abundance of food in the medium.

Suppose that a population of bacteria is growing exponentially and that it doubles in 40 minutes. If there are 10^6 bacteria initially, how many bacteria are present in 60 minutes?

Solution:

Measure time in minutes.
The model is $P(t) = P(0)e^{kt}$ and we are given that

$$\frac{P(40)}{P(0)} = 2 = e^{k40}$$

We can solve this for k to get $40k = \log 2$. Then

$$P(t) = P(0)e^{kt} = 10^6 e^{\frac{t \log 2}{40}}$$

We want $P(60)$.

$$P(60) = 10^6 e^{\frac{60 \log 2}{40}} = 10^6 2^{\frac{3}{2}} = 2.83 \times 10^6$$

3. A population of bacteria is growing under ideal laboratory conditions. Thus the population is growing exponentially. Suppose that after 4 hours there are 5000 bacteria while after 12 hours there are 10,000 bacteria. How many bacteria cells were present initially?

Solution:

We are told that the model to use is $P(t) = P(0)e^{kt}$. We want $P(0)$. We are also told that

$$P(4) = P(0)e^{4k} = 5{,}000$$
$$P(12) = P(0)e^{12k} = 10{,}000$$

This is two equations in two unknowns.

Divide the second by the first to cancel $P(0)$:

$$e^{12k-4k} = 2$$

So $e^{8k} = 2$ and we can solve by getting a value for k, or by getting a value for e^k

or e^{4k}. Lets take the last option: from $e^{8k} = 2$ we get $e^{4k} = \sqrt{2}$.

$$P(0)e^{4k} = 5{,}000$$

Then we get $P(0) = \frac{5000}{\sqrt{2}} = 3535$ bacteria. We could have solved for k via $\log e^{8k} = \log 2$.

Summary

1. Exponential models are described either by:

$$P(t) = P(0)e^{kt}$$

or

$$y' = ky.$$

2. Any one of the four quantities can be an unknown in a problem.

3. There are many examples of exponential models in science, finance, engineering, and economics.

4. Facility in working with exponential equations is strong asset.

Note:

Exponential Models II – looks at radioactive decay problems

Exponential Models III – looks at financial models and Newton cooling.