Mathematics Tutorial Series

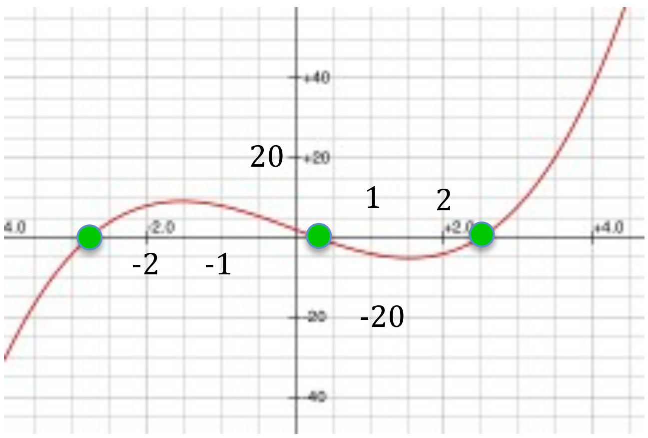
Applications of Calculus

Newton's Method of Finding Roots

Suppose we want to find the values of x that make

$$x^3 - 7x + 2 = 0$$

These are the "roots" of the cubic polynomial.



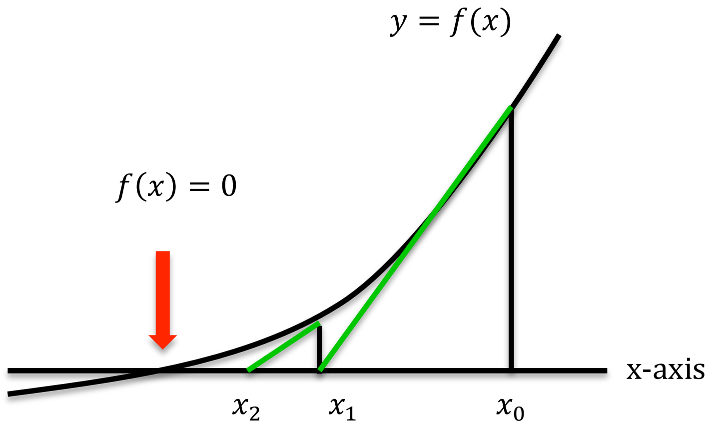
There are roots near: 2.5, 0.3 and -2.8.

You can change other problems into "root" problems.

To find x such that $x = 8\sin x$ solve $x - 8\sin x = 0$

To calculate $\sqrt{117}$ solve $x^2 - 117 = 0$

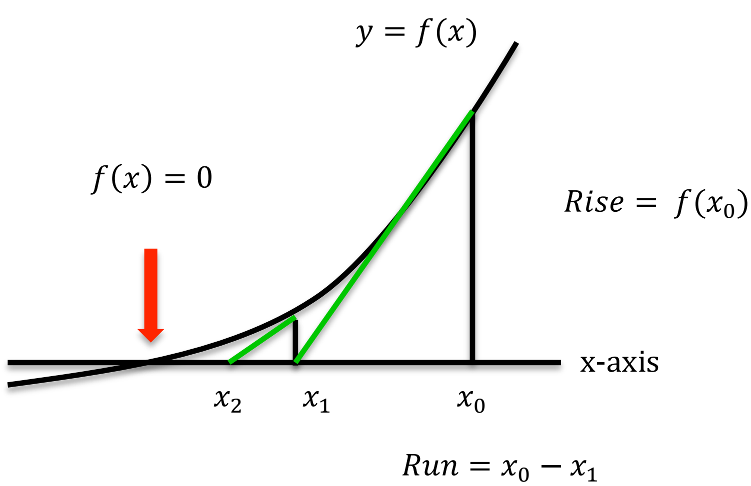
The Idea of Newton's Method



The method starts with an educated guess x_0
Find the point above x_0 on the graph $(x_0, f(x_0))$.
Follow the tangent at this point back to the x-axis.
The tangent hits the x-axis at our next estimate x_1.

Repeat

Details



Slope of the tangent = $\frac{Rise}{Run} = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$

So $\frac{f(x_0)}{f'(x_0)} = x_0 - x_1$ hence $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

In general:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

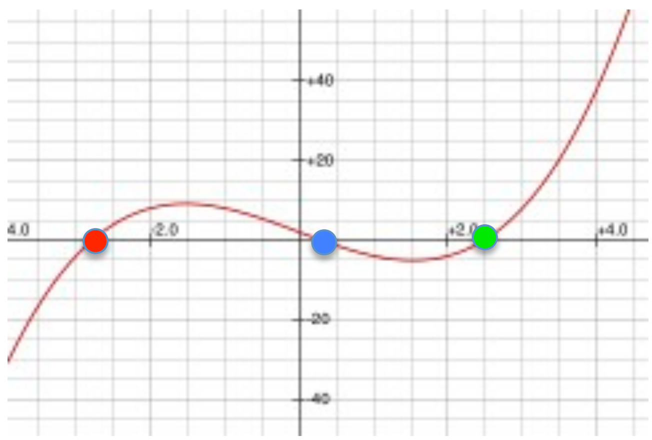
$f(x) = x^3 - 7\,x + 2$

$f'(x) = 3x^2 - 7$

$$x_{n+1} = x_n - \frac{x_n^3 - 7\,x_n + 2}{3\,x_n^2 - 7}$$

Use a spreadsheet!

n			
0	0	5	-2
1	0.2857	3.6471	-3.6
2	0.2892	2.8879	-2.9897
3	0.2892	2.5622	-2.7982
4	0.2892	2.4925	-2.7787
5	0.2892	2.4893	-2.7785



Example 2

Set up Newton's Method to find the roots of:

$$y = x^3 - 7$$

<u>Solution</u>

$$y = x^3 - 7$$

$$y' = 3x^2$$

So the function we iterate is:

$$x_{n+1} = x_n - \frac{x_n^3 - 7}{3x_n^2}$$

$$x \leftarrow \frac{2x^3 + 7}{3x^2}$$

Start at $x_0 = 2$, say.

x_0	2.000000000
x_1	1.916666667
x_2	1.912938458
x_3	1.912931183
x_4	1.912931183
x_5	1.912931183
x_6	1.912931183
X_7	1.912931183

Newton's Method doesn't always work well

Sometimes it doesn't work at all

Basic questions:

1. How do we pick a starting point?
2. How many iterations should we use?
3. How can we tell if the method is working?
4. What kinds of things can go wrong?

Example 3

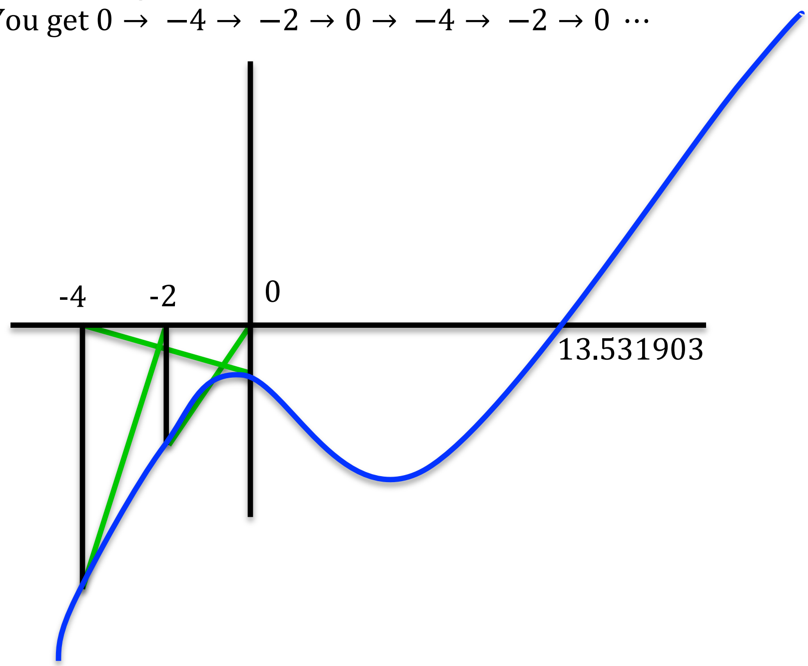
$$x^3 - 12\,x^2 - 16\,x - 64$$

$$y' = 3\,x^2 - 24\,x - 16$$

$$x_{n+1} = x_n - \frac{{x_n}^3 - 12\,{x_n}^2 - 16\,x_n - 64}{3\,{x_n}^2 - 24\,x_n - 16}$$

The polynomial $x^3 - 12\,x^2 - 16\,x - 64$ has one real root.
It is approximately 13.531903.

Try starting at $x_0 = 0$.
You get $0 \rightarrow\ -4 \rightarrow\ -2 \rightarrow 0 \rightarrow\ -4 \rightarrow\ -2 \rightarrow 0\ \cdots$

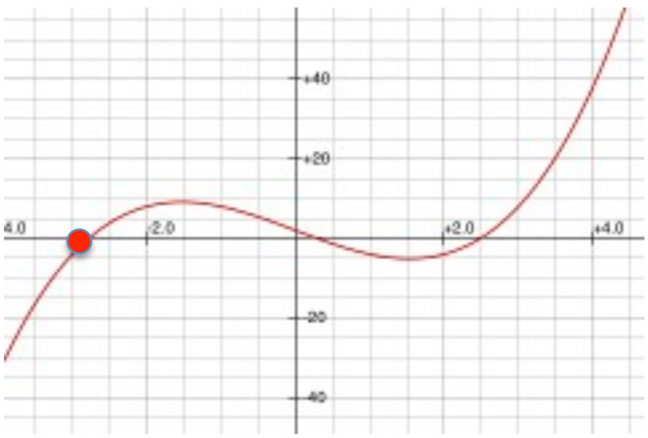


Example 4

$f(x) = x^3 - 7x + 2$

$f'(x) = 3x^2 - 7$

$$x_{n+1} = x_n - \frac{x_n^3 - 7x_n + 2}{3x_n^2 - 7}$$



Start at $x_0 = -1.1275$

x_0	-1.1275
x_1	+1.5274
x_2	- 4872.8052
x_3	- 3248.5371
⋮	⋮
After many steps	

I started at a special point.
Then went near the local minimum.
Here the tangent is almost horizontal.
So the next iteration is way off scale left.
The process then slowly works its way back to the red root.

Summary

1. Newton's Method of finding roots is an iterative process
2. It moves vertically to the curve then follows the tangent back to the x-axis.
3. Use a graph of the function to understand where to start and how quickly the sequence converges
4. There is a lot that can be proved about the convergence of the method.
5. It provides a door into the study of numerical analysis
6. It is one of the paths into the study of fractals.
7. Many people contributed to the development of the method; Newton was only one of them.

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March 2012